Spherical designs and some generalizations: An overview

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Spherical $t$-designs

$S^{n-1} = \{(x_1, x_2, \cdots, x_n) \in \mathbb{R}^n \mid x_1^2 + x_2^2 + \cdots + x_n^2 = 1\}$. $Y \subset S^{n-1}$, $0 < |Y| < \infty$.

Definition (Delsarte-Goethals-Seidel, 1977)

$Y$ is called a spherical $t$-design on $S^{n-1}$ if and only if

$$
\frac{1}{|S^{n-1}|} \int_{S^{n-1}} f(x) d\sigma(x) = \frac{1}{|Y|} \sum_{x \in Y} f(x)
$$

for any polynomials $f(x) = f(x_1, x_2, \cdots, x_n)$ of degree $\leq t$. 
Some generalizations

(1). Replacing the sphere $S^{n-1}$ with other spaces.

(i) compact symmetric spaces of rank 1

($=\text{Projective spaces over }\mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}$)

(Further generalization: Designs on Grassmannian spaces, cf. talk of Martin Ehler)

(ii) $\binom{V}{k} = \text{Johnson } J(v, k) \text{ association scheme}

(iii) $\mathbb{F}_2^n = \text{Hamming association scheme } H(n, 2)$

(or Q-polynomial association schemes)

(2). Replacing the space of “Polynomials of degree at most $t$” by other set of functions

But I will not talk much about these in this talk.
Spherical $t$-designs

(I)

allow several concentric spheres

(=Euclidean $t$-designs)

(III)

weighted spherical $t$-designs on $S^{n-1}$

(=cubature formula of degree $t$)

(II)
(II) weighted spherical $t$-designs in $S^{n-1}$

(=cubature formula of degree $t$ on $S^{n-1}$)

$Y \subset S^{n-1}$, $0 < |Y| < \infty$

$w : Y \rightarrow \mathbb{R}_{>0}$

$(Y, w)$ is a weighted spherical $t$-design on $S^{n-1}$

$\iff$

$$\frac{1}{|S^{n-1}|} \int_{S^{n-1}} f(x) d\sigma(x) = \sum_{y \in Y} w(y) f(y),$$

for $\forall f(x) = f(x_1, \ldots, x_n)$, polynomial of degree $\leq t$
(III) allow several concentric spheres

(=Euclidean $t$-designs)

$Y \subset S^{n-1}(r_1) \cup S^{n-1}(r_2) \cdots \cup S^{n-1}(r_p)$

$0 < |Y| < \infty$

$w : Y \longrightarrow \mathbb{R}_{>0}$

**Definition:**

$(Y, w)$ is a Euclidean $t$-design

$\iff$

$$\sum_{i=1}^{p} \frac{w(Y \cap S^{n-1}(r_i))}{|S^{n-1}(r_i)|} \int_{S^{n-1}(r_i)} f(x) d\sigma(x) = \sum_{y \in Y} w(y) f(y),$$

$\forall f(x) = \text{polynomial of degree} \leq t$
Fisher type inequalities

(I), (II)

\( Y = t\)-design or

\((Y, w)\)=weighted \( t\)-design \( \implies |Y| \geq \)

\[
\begin{cases}
\text{if } t = 2e, \\
\left(\binom{n-1+e}{e}\right) + \left(\binom{n-1+e-1}{e-1}\right) = m_e,
\end{cases}
\]

with \( m_e = m_e + m_{e-1} + \cdots + m_1 + m_0 \)

\( m_i = \dim(Harm_i(\mathbb{R}^n)) \)

\[
= \binom{n-1+i}{i} - \binom{n-1+i-2}{i-2}
\]

if \( t = 2e + 1 \)

\[
2\binom{n-1+e}{e}
\]

(III)

\((Y, w) = \)

Euclidean \( t\)-design \( \implies |Y| \geq \)

\[
\begin{cases}
\text{if } t = 2e, \\
\overline{m_e} + \overline{m_{e-1}} + \cdots + \overline{m_{e-p+1}}
\end{cases}
\]

if \( t = 2e + 1 \)

more complicated
Tight designs: Those satisfying one of the equality in above

(I) tight spherical $t$-designs

Delsarte-Goethals-Seidel (1977)
Bannai-Sloane (1981)
Nebe-Venkov (2012)
still open for $t = 4, 5, 7$

(II) tight weighted spherical $t$-designs

$\implies$ tight spherical $t$-design
( reduced to (I) )
(III) Classification problems of tight Euclidean $t$-designs

Neumaier-Seidel (1988)
Delsarte-Seidel (1989)
Bajnok (2006)
B-B (2005 $\sim$ ...)
B-B-Hirao-Sawa (2010)
many papers
B-B (2014) Moscow J. of Combinatorics and Number Theory
mostly on two shells (spheres)

(The results are still very partial !)
Another generalization

(IV) spherical designs of harmonic index $t$

$Y \subset S^{n-1}, \ 0 < |Y| < \infty,$

\[ \sum_{y \in Y} f(y) = 0, \ \text{for} \ \forall f(y) \in Harm_t(\mathbb{R}^n). \]

(Note that $Y$ is a spherical $t$-design

\[ \iff \sum_{y \in Y} f(y) = 0, \ \text{for} \ \forall f(y) \in Harm_i(\mathbb{R}^n) \ \text{and} \ 1 \leq i \leq t. \)
Fisher type in equality for (IV)

\[ Q_i(x) = \text{Gegenbauer polynomial of degree } i. \]

i.e. \( \int_{-1}^{1} Q_i(x)Q_j(x)(1 - x^2)^{n-3} = \alpha_i\delta_{i,j}, \alpha_i > 0 \)

\[ Q_i(1) = \binom{n-1+i}{i} - \binom{n-1+i-2}{i-2} = \dim(Harm_i(\mathbb{R}^n)) \]

\[ c_{2e} = -\min. \text{ value of } Q_{2e}(x) \]

\[ Q_0(x) = 1 \]
\[ Q_1(x) = nx \]
\[ Q_2(x) = \frac{n+2}{2}(nx^2 - 1) \]
\[ Q_3(x) = \frac{n(n+4)}{3!}((n + 2)x^2 - 3) \]
\[ Q_4(x) = \frac{n(n+6)}{4!}((n + 4)(n + 2)x^4 - 6(n + 2)x^2 + 3) \]
(The graph of $y = Q_8(x)$)
$Y$ = a spherical designs of harmonic index $t$.

Let $A(n, t) =$ smallest $|Y|$ such that $Y$ is a spherical
design of harmonic index $t$ on $S^{n-1}$.

(1) $t = 2e + 1 \implies |Y| \geq 2$, since $Y = \{y, -y\}$ is a
design of harmonic index $2e + 1$ on $S^{n-1}$
So, $A(n, 2e + 1) = 2$

(2) $t = 2 \implies |Y| \geq n$, $Y = \{e_1, e_2, \ldots, e_n\}$
is a design of harmonic index 2 on $S^{n-1}$

(3) $n = 2$ ($t = 2e$) $\implies |Y| \geq 2$, $x, y \in S^1$ with the angle
$\theta = \pi/2e$ is such an example
So, $A(2, 2e) = 2$ for $\forall e$
First nontrivial case: $A(3, 4)$

Let $m_i = 2i + 1$, and let $f_1, f_2, \ldots, f_{2i+1}$ be an orthonormal basis of $Harm_i(R^3)$. Our case is $i = 4$. Then minimize

$$\left\{ \sum_{j=1}^{9} \left( \sum_{y \in Y} f_j(y) \right)^2 \mid Y \subset S^2, |Y| \text{ is fixed} \right\}$$

(Note that this quantity is $= 0$, if and only if $Y$ is a design of harmonic index 4 on $S^2$.)

$|Y| = 2, 3, 4$ are shown to be impossible.

$|Y| = 5$. Examples were found by numerical experiments. (Namely, we obtained regular pentagons of two different sizes.)
Theorem (B-Okuda-Tagami)

\[ |Y| = \text{a spherical } 2e \text{ -design on } S^{n-2}, \]
\[ r = \text{a root of } Q_{2e}(x), \]
Let \( Y' \subset S^{n-2} \) be defined by

\[ Y' = \{(r, \sqrt{1 - r^2} \, x) \in S^{n-1} \mid x \in Y\}. \]

Then \( Y' \) is a design of harmonic index \( 2e \) on \( S^{n-1} \)

Theorem (B-O-T) \[ A(3, 4) = 5. \]

Moreover, if \( Y \) is a design of harmonic index 4 on \( S^2(\subset \mathbb{R}^3) \),
and \( |Y| = 5 \). Then \( Y \) is a pentagon on the circle of
radius \( \sqrt{1 - r^2} \), where \( r \) is a zero of \( Q_4(x) \) (for \( n = 3 \)), i.e.
\[ r_1, \ r_2 = \frac{1}{35} \sqrt{525 + 70\sqrt{30}} \]
or those with any points replaced by their antipodal points.
Remark. $5 \leq A(3, 4) \leq 6$. (A half of the 12 vertices of a regular icosahedron gives such an example.

We want to determine $A(n, t)$ for general $n$ and $t$, but most cases are open.
Let
\[ c_t = -\min\{Q_t(x) \mid -1 \leq x \leq 1\}. \]

(So, \( c_t > 0 \).)

Apply Delsarte’s method (LP-method) to
\[ F(s) = c_t + Q_t(s) \text{ on } -1 \leq s \leq 1. \]
(Note that \( F(s) \geq 0 \), for \(-1 \leq s \leq 1\).)

Then we get

**Theorem (B-O-T)** If \( Y \) is a design of harmonic index \( t \) on \( S^{n-1} \), then
\[ |Y| \geq b_t(= b_{t,n}) := 1 + \frac{Q_t(1)}{c_t}. \]
Proof. Calculate $\sum_{(x,y)\in Y \times Y} F(x \cdot y)$ in two ways.

(i) \[
\sum_{(x,y)\in Y \times Y} F(x \cdot y) = \sum_{(x,y)\in Y \times Y} (c_t + Q_t(x \cdot y)) = c_t \cdot |Y|^2
\]
(Since $\sum_{(x,y)\in Y \times Y} Q_t(x \cdot y) = 0$ if $Y$ is a design of harmonic index $t$.)

(ii) \[
\sum_{(x,y)\in Y \times Y} F(x \cdot y) \geq |Y| \cdot F(1).
\]
(Since $F(s) \geq 0$ on $-1 \leq s \leq 0$.)

So, $c_t \cdot |Y|^2 \geq |Y| \cdot F(1)$. Hence

\[
|Y| \geq 1 + \frac{Q_t(1)}{c_t} (= b_t).
\]
In the above proof, note that, by using ”Addition formula”,

\[
\sum_{(x,y) \in Y \times Y} Q_i(x \cdot y) = \sum_{(x,y) \in Y \times Y} (\sum_{j=1}^{m_i} f_j(x)f_j(y)) = \sum_{j=1}^{m_i} (\sum_{y \in Y} f_j(x))^2 \geq 0.
\]

and ” = 0”, if and only if \(\sum_{y \in Y} f(y) = 0\), for \(\forall f(x) \in Harm_i(R^n)\).

(So, \(\sum_{(x,y) \in Y \times Y} Q_t(x \cdot y) = 0\) if and only if \(Y\) is a design of harmonic index \(i\).)

Here, \(m_j = \text{dim}(Harm_i(R^n))\), and \(\{f_1, f_2, \ldots, f_{m_i}\}\) is any orthonormal basis of \(Harm_i(R^n)\).
Values of $b_t = b_{t,n}$ for designs of harmonic index $t$ on $S^{n-1}$

<table>
<thead>
<tr>
<th></th>
<th>$n = 3$</th>
<th>$n = 4$</th>
<th>$n = 5$</th>
<th>$n = 6$</th>
<th>$n = 7$</th>
<th>$n = 8$</th>
<th>$n = 9$</th>
<th>$n = 10$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$t = 4$</td>
<td>3.33...</td>
<td>5</td>
<td>7</td>
<td>9.33...</td>
<td>12</td>
<td>15</td>
<td>18.33...</td>
<td>22</td>
</tr>
<tr>
<td>$t = 6$</td>
<td>3.41...</td>
<td>5.29...</td>
<td>7.69...</td>
<td>10.69...</td>
<td>14.33...</td>
<td>18.67...</td>
<td>23.76...</td>
<td>29.68...</td>
</tr>
<tr>
<td>$t = 8$</td>
<td>3.44...</td>
<td>5.41...</td>
<td>8.01...</td>
<td>11.35...</td>
<td>15.55...</td>
<td>20.72...</td>
<td>27.004...</td>
<td>34.52...</td>
</tr>
<tr>
<td>$t = 10$</td>
<td>3.45...</td>
<td>5.47...</td>
<td>8.18...</td>
<td>11.73...</td>
<td>16.26...</td>
<td>21.97...</td>
<td>29.04...</td>
<td>37.69...</td>
</tr>
<tr>
<td>$t = 12$</td>
<td>3.46...</td>
<td>5.51...</td>
<td>8.28...</td>
<td>11.95...</td>
<td>16.71...</td>
<td>22.77...</td>
<td>30.39...</td>
<td>39.84...</td>
</tr>
<tr>
<td>$t = 14$</td>
<td>3.46...</td>
<td>5.53...</td>
<td>8.35...</td>
<td>12.10...</td>
<td>17.22...</td>
<td>23.32...</td>
<td>31.33...</td>
<td>41.37...</td>
</tr>
<tr>
<td>$t = 16$</td>
<td>3.47...</td>
<td>5.54...</td>
<td>8.39...</td>
<td>12.21...</td>
<td>17.37...</td>
<td>23.71...</td>
<td>32.01...</td>
<td>42.48...</td>
</tr>
<tr>
<td>$t = 18$</td>
<td>3.47...</td>
<td>5.56...</td>
<td>8.42...</td>
<td>12.28...</td>
<td>17.37...</td>
<td>24.004...</td>
<td>32.51...</td>
<td>43.32...</td>
</tr>
<tr>
<td>$t = 20$</td>
<td>3.47...</td>
<td>5.56...</td>
<td>8.45...</td>
<td>12.34...</td>
<td>17.49...</td>
<td>24.22...</td>
<td>32.89...</td>
<td>43.97...</td>
</tr>
</tbody>
</table>
Spherical designs of harmonic index 4.

Let $Y$ be a design of harmonic index 4 on $S^{n-1}$. Then

$$|Y| \geq \frac{(n + 1)(n + 2)}{6}.$$  

If $|Y| = \frac{(n + 1)(n + 2)}{6}$, then $I(Y) = \{ \pm \alpha \}$ with $\alpha = \sqrt{\frac{3}{n+4}}$ where $Q_4(\alpha)$ gives the minimum value of $Q_4(x)$.

So, $Y$ gives a set of equiangular lines ($\frac{(n + 1)(n + 2)}{6}$ lines) with inner product $\alpha = \sqrt{\frac{3}{n+4}}$.

Okuda-Yu (2015) showed that there exist no such set of equiangular lines (by using SDP-method). So, there is no tight spherical designs of harmonic index 4 if $n \geq 3$. (Cf. Wei-Hsuan Yu’s talk.)
What happens if $t = 2e$ is fixed, and $n \to \infty$?

$t = 4 \iff b_4 = \frac{(n+1)(n+2)}{6} \approx \frac{1}{6}n^2$.

$t = 6 \iff b_6 \approx \frac{1}{20(2+\sqrt{10})}n^3$.

$t = 8 \iff b_8 \approx \frac{1}{(\approx 3,000)}n^4$.

\vdots

$t = 2e \iff b_{2e} \approx (\text{const})n^e$.

(Note that this constant is explicitly calculable.)

If $Y$ is a tight design of harmonic index $2e$, then $I(Y) = \{\pm \alpha\}$, and so, $|Y| \leq \frac{n(n+1)}{2}$ (absolute bound).

So, there are no tight designs of harmonic index $2e$, if $e \geq 3$, in general.
What happens for $b_{2e,n}$, if $n$ is fixed and $e \to \infty$?

$$\lim_{e \to \infty} b_{2e,n} \text{ are, for } n = 3, 4, \ldots, 10:$$

3.428 (6), 5.079 (10), 8.559 (15), 16.426 (21) 35.118 (38),
81.85 (36), 204.5 (45) 541.65 (55), (respectively),
where inside the parentheses denote the value $\frac{n(n+1)}{2}$.

So, $\lim_{e \to \infty} b_{2e,n}$ is far bigger than $\frac{n(n+1)}{2}$, except for some small $n$, say $n \leq 6$.

It would be a very interesting open question, what are the values of $A(3,2e)$ (or $A(4,2e)$) for large $e$.
This implies that tight designs of harmonic index $2e$ do not exist in general (i.e. for large $t$), unless $n$ is very small.
Let $T = \{t_1, t_2 \ldots, t_l\}$ be a set of natural numbers. $Y \subset S^{n-1}$, $0 < |Y| < \infty$ is called a spherical design of harmonic index $T$, if

$$\sum_{y \in Y} f(y) = 0 \text{ for } \forall f(x) \in Harm_i(\mathbb{R}^n) \text{ with } \forall i \in T.$$ 

(Spherical $t$-design is a design of harmonic index $\{1, 2, \ldots, t\}$.)

$t = 8$

- 8-design
- 8-design
- 8-design
- 8-design
- 8-design

$|Y| \geq \approx \frac{1}{24} n^4$

$|Y| \geq \approx \frac{1}{168} n^4$ for $\{2, 8\}$

$|Y| \geq \approx \frac{1}{252} n^4$ for $\{4, 8\}$
(V) spherical null $t$-design

Let $Y \subset S^{n-1}$, $0 < |Y| < \infty$

$w : Y \rightarrow \mathbb{R}_{\neq 0}$

$Y$ is called a spherical null $t$-design

$\iff \sum_{y \in Y} w(y) f(y) = 0$ for $\forall f(x) = f(x_1, \ldots, x_n)$ of degree $\leq t$

$\iff \sum_{y \in Y} w(y) f(y) = 0$ for $\forall f(x) \in Harm_i(\mathbb{R}^n)$ with $i = 0, 1, \ldots, t$
What are Fisher type lower bound for null spherical $t$-designs on $S^{n-1}$? (working with Francis Campena)

Let

$$A(n, t) = \min\{|Y| \mid Y \text{ is a null spherical } t\text{-design}\}$$

What are $A(n, t)$?

**Conjecture** $A(n, t) = 2(t + 1), \forall n \geq 2, \forall t$

**Partial results**

$A(n, t) \leq (t + 1), \forall n \geq 2, \forall t$

$A(2, t) = (t + 1), \forall t$

$A(3, 2) = 6,$

$A(n, 2) = 6, \forall n \geq 2$

(First open case $A(3, 3), 7 \leq A(3, 3) \leq 8$)
Replace $S^{n-1}$ by association schemes

(II) weighted combinatorial $t$-designs
   (cf. talks of Da Zhao and Zongchen Chen)

(I)
combinatorial $t$-designs
designs in $J(v, k)$

(III) relative $t$-designs on $J(v, k)$
   (Delsarte (1977), Pairs of vectors .......)
   (cf. talk of Yan Zhu)

(II) weighted $t$-designs in $H(n, 2)$

(I)
t-designs in $H(n, 2)$

(III) relative $t$-designs on $H(n, 2)$
   (Delsarte (1977))
   (=weighted regular $t$-wise balanced designs)
   (cf. talks by Yan Zhu and Lihang Hou)
Fisher type lower bounds

(I) (II) \implies |Y| \geq \overline{m}_e, \text{ for } t = 2e \text{ where}

\overline{m}_i = m_i + m_{i-1} + \cdots + m_1 + m_0

m_i = \text{rank}(E_i) = \begin{cases} \binom{v}{i} - \binom{v}{i-1} & \text{for } J(v, k) \\ \binom{n}{i} & \text{for } H(n, 2) \end{cases}

(III) \implies |Y| \geq \overline{m}_e + \overline{m}_{e-1} + \cdots \overline{m}_{e-p+1} \text{ for } t = 2e

study of tight relative \( t \)-designs

on \( H(n, q) \) and \( J(v, k) \)

(cf. talks by Yan Zhu and Lihang Hou)
Thank You