Synchronizing automata: new techniques and results

Raphaël Jungers

UCLouvain

Jiao Tong Univ., Apr. 2015

Joint work with
François Gonze
Synchronizing automata

Definition: A (complete deterministic) automaton is synchronizing if there is a sequence of colors such that all the paths compatible with this sequence end in the same node.

Cerny’s conjecture (1964): If a graph is synchronizing, then it admits a synchronizing sequence of length at most \((n-1)^2\).

Connected with the Road coloring conjecture [1977 Adler et al.] [2007 Trahtman]
Disclaimer

What I won’t do today

- Prove Cerny’s conjecture in particular cases
- Improve the upper bound on the shortest synchronizing word
  (though I would love to!)

But…

Develop new tools
Proof of concept
Mostly ideas, very few technical considerations
Plan

- Černý’s conjecture
- Approach: the triple rendezvous time
- A tool: the synchronizing probability function
- Counter examples
Outline

- Synchronizing automata, Cerny’s conjecture, and previous approaches
- The synchronizing probability function and previous results
- New results: a counterexample and a new upper bound (on a related quantity)
- Discussion
Outline

- Synchronizing automata, Cerny’s conjecture, and previous approaches

- The synchronizing probability function and previous results

- New results: a counterexample and a new upper bound (on a related quantity)

- Discussion
Theorem [1990 Eppstein]: Synchronizing graphs are recognizable in polynomial time.

Length \((n-1)^2\)
Previous approaches (1)

Cerny’s conjecture (1964): If a graph is synchronizing, then it admits a synchronizing sequence of length at most \((n-1)^2\)

Known upper bounds on the shortest synchronizing word:

- [1964 Cerny] \(2^n\)
- [1966 Starke] \(\frac{n^3}{2} - \frac{3}{2} n^2 + n + 1\)
- [1970 Kohavi] \(\frac{n(n-1)^2}{2}\)
- [1978 Pin] \(\frac{7}{27} n^3 - \frac{17}{18} n^2 + \frac{17}{6} n - 3\)
- [1982 Frankl (Pin)] \(\frac{n^3-n}{6}\)
  - The best so far!

[Gonze, Trahtman, J. 2015]
Previous approaches (2)

**Cerny’s conjecture (1964):** If a graph is synchronizing, then it admits a synchronizing sequence of length at most \((n-1)^2\).

- **Particular cases**
  - [1981 Pin] small rank \((\log(n))\), circular of prime size
  - [1990 Eppstein] monotonic
  - [1998 Dubuc] circular
  - [2001 Kari] Eulerian
  - [2009 Trahtman] aperiodic
  - [2009 Beal Perrin] one-cluster
  - [2009 Carpi d’Alessandro] locally strongly transitive
  - [2009 Volkov] partial order-related
  - [2010 Steinberg] ...

- **Complexity issues**
  - NP-hard [1990 Eppstein]
  - Apx-hard [2010 Berlinkov]
Outline

• Synchronizing automata, Cerny’s conjecture, and previous approaches

• The synchronizing probability function and previous results

• New results: a counterexample and a new upper bound (on a related quantity)

• Discussion
Synchronizing automata

We need a more holistic approach

Theorem [1990 Eppstein]: Synchronizing graphs are Recognizable in polynomial time.

Eppstein’s square graph gives a poor strategy to find a short synchronizing word

→ motivation: take into account the whole set of color sequences of a length $t$, not only the best one
A simple game

- Two players playing on a graph: the «mouse» and the «cat»

- A parameter \(t\) (here, \(t=2\))

- The cat is hidden somewhere on a colored graph, and the mouse must pick up a node where to catch him

- Before to do that, the mouse may impose the cat to follow a particular sequence of colors of length \(t\)

- The cat wants to minimize the probability to get caught
Two players playing on a graph: the « mouse » and the « cat »

A parameter t (here, t=2)

Definition: The synchronizing probability function $k(t)$ of the automaton is the smallest probability the cat can ensure to get caught, whatever strategy (of length t) the mouse chooses.
The synchronizing probability function

- The cat’s strategy must be probabilistic (i.e. a probability function on the nodes)

**Definition:** The synchronizing probability function $k(t)$ of the automaton is the smallest probability the cat can ensure to get caught, whatever strategy (of length $t$) the mouse chooses
The cat’s strategy must be probabilistic (i.e. a probability function on the nodes)

Definition: The synchronizing probability function $k(t)$ of the automaton is the smallest probability the cat can ensure to get caught, whatever strategy (of length $t$) the mouse chooses.
The synchronizing probability function

- The cat’s strategy must be probabilistic (i.e. a probability function on the nodes)
  \( k(0) = \frac{1}{2} \)
  \( k(1) = 1 \)

- Definition: The synchronizing probability function \( k(t) \) of the automaton is the smallest probability the cat can ensure to get caught, whatever strategy (of length \( t \)) the mouse chooses.
  \( p(1) = \frac{1}{2} \)
  \( p(2) = \frac{1}{2} \)
  \( k(1) = 1 \)
The synchronizing probability function

- The cat’s strategy must be probabilistic (i.e. a probability function on the nodes)
  
  \[ k(0) = \frac{1}{2} \]

  \[ k(1) = 1 \]

- Definition: The synchronizing probability function \( k(t) \) of the automaton is the smallest probability the cat can ensure to get caught, whatever strategy (of length \( t \)) the mouse chooses.
The synchronizing probability function

- The cat’s strategy must be probabilistic (i.e. a probability function on the nodes)
  
  \[ k(0) = \frac{1}{2} \]

  \[ k(1) = 1 \]

Definition: The synchronizing probability function \( k(t) \) of the automaton is the smallest probability the cat can ensure to get caught, whatever strategy (of length \( t \)) the mouse chooses.
The synchronizing probability function

- **Proposition:** The automaton has a synchronizing word of length $t$ if and only if $k(t)=1$

- Thus Cerny’s conjecture is:

$$k((n-1)^2)=1$$

- Note that in general, the *mouse’s policy* might be probabilistic as well
A few equations…

- **Definition:** The synchronizing probability function $k(t)$ of the automaton is the smallest probability the cat can ensure to get caught, whatever strategy (of length $t$) the mouse chooses.

$$\min_{p,k} k$$

$$s.t. \quad pA \leq ke^T \quad \forall A \in \Sigma^{\leq t}$$

$$ep^T = 1$$

$$p \geq 0.$$ 

- The problem he has to solve is an LP (Linear Program)!
A few equations…

- **Definition**: The synchronizing probability function $k(t)$ of the automaton is the **smallest probability** the cat can ensure to get caught, whatever strategy (of length $t$) the mouse chooses.

\[
\begin{align*}
\min_{p,k} & \quad k \\
\text{s.t.} & \quad pA \leq ke^T \\
& \quad ep^T = 1 \\
& \quad p \geq 0.
\end{align*}
\]

\[
\begin{align*}
\max_{q,k} & \quad k \\
\text{s.t.} & \quad Aq \geq ke^T \\
& \quad eq = 1 \\
& \quad q \geq 0.
\end{align*}
\]

- The problem he has to solve is an **LP** (Linear Program)!
The synchronizing function on practical examples

- Cerny’s automaton
The synchronizing function on practical examples

• Kari’s automaton
The synchronizing function on practical examples

- Roman’s automaton
A few first results

- **Theorem:** The players can communicate their policies

- A procedure allowing to compute the function pretty fast in practice

- **Proposition:** It doesn’t help the mouse to allow her to take shorter products

- **Proposition:** there is always an optimal policy for the mouse with at most \( n \) different columns (\( n \) is the number of nodes)

- **Theorem:** If \( k(t)<1 \), then \( k(t+(n-1))>k(t) \)

- Means « \( k(t) \) cannot stagnate too long »
Definition: The synchronizing probability function $k(t)$ of the automaton is the smallest probability the cat can ensure to get caught, whatever strategy (of length $t$) the mouse chooses.

\[
\min_{p,k} k \\
\text{s.t. } pA \leq k e^T \quad \forall A \in \Sigma^{\leq t} \\
e p^T = 1 \\
p \geq 0.
\]

The problem he has to solve is an LP (Linear Program)!
A few first results

- **Theorem:** The players can communicate their policies.

- A procedure allowing to compute the function pretty fast in practice.

- **Proposition:** It doesn’t help the mouse to allow her to take shorter products.

- **Proposition:** There is always an optimal policy for the mouse with at most \( n \) different rows (\( n \) is the number of nodes).

- **Theorem:** If \( k(t) < 1 \), then \( k(t + (n-1)) > k(t) \).

  - Means « \( k(t) \) cannot stagnate too long ».
The synchronizing function on practical examples

- Cerny’s automaton

Theorem: If $k(t) < 1$, then $k(t + (n-1)) > k(t)$

- Means « $k(t)$ cannot stagnate too long »
Proof of the theorem

**Theorem:** If \( k(t) < 1 \), then \( k(t + (n-1)) > k(t) \)

**Proof:** suppose \( k(t) = k(t+1) \)

- Look at the polytope \( P_t \) of optimal solutions
  \[
  P_t = \{ p : A_0 A_1 A_2 \ldots A_{t-1} p \leq k \}
  \]

- **Lemma:** \( P_{t+1}' \) is in \( P_t' \)

- **Lemma:** \( P_{t+1}' \) is different from \( P_t' \)
  **Proof:** if not, then \( P_{t+2}' = P_{t+1}' \)

- **Lemma:** This implies that \( \text{dim } P_{t+1}' < \text{dim } P_t' \)

- Since \( \text{dim } P_t < n-1 \), after at most \( n-1 \) steps it cannot decrease anymore
A conjecture

• **Observation:** At some fixed times, the value of the function is always higher (or equal) than Černý’s automaton

\[ t = 1 + (n+1) \cdot i \]

• **Conjecture:** It is always the case

For any synchronizing automaton and for any \( j \geq 1, j \leq n - 1, \)

\[ k(1 + (j - 1)(n + 1)) \geq j/(n - 1). \]

This conjecture is stronger than Černý’s conjecture.
For a synchronizing automaton $\Sigma$, the *triple rendezvous time* $T_{3,\Sigma}$ is the length of the shortest word mapping three states to a single one.
Conjectures

Conjecture:

For any synchronizing automaton and for any \( j \geq 1, j \leq n - 1, \)

\[
k(1 + (j - 1)(n + 1)) \geq j / (n - 1).
\]

An easier conjecture on the triple rendezvous time:

For any synchronizing automaton, \( T_3 \leq n + 2. \)
Outline

• Synchronizing automata, Cerny’s conjecture, and previous approaches

• The synchronizing probability function and previous results

• New results:
  – a new upper bound (on a related quantity)
  – a counterexample

• Discussion
First bound on the TRT

For any synchronizing automaton,

Each pair of states can be mapped to a single state with a word of length at most $\frac{n(n-1)}{2}$.

In a synchronizing automaton $\Sigma$ with $n$ states,

$T_{3,\Sigma} \leq \frac{n(n-1)}{2} + 1$. 

A better bound using the SPF

We obtain a better bound on the triple rendezvous time:

\[ T_3 \leq \sum_{s=0}^{n-1} (\lfloor (n - s)/2 \rfloor + 1) = \frac{n(n + 4)}{4} - \frac{n \mod(2)}{4}. \]
A better bound using the SPF

First observation: we can represent $A(t)$ on a graph!

Example for $t=1$: 
A better bound using the SPF

First observation: we can represent $A(t)$ on a graph!
A better bound using the SPF

First observation: we can represent $A(t)$ on a graph!

**Lemma:** There is always an optimal solution for the mouse with a disconnected union of singletons, pairs, and odd cycles.
A better bound using the SPF

First observation: we can represent $A(t)$ on a graph!

Lemma: There is always an optimal solution for the mouse with a disconnected union of singletons, pairs, and odd cycles.
A better bound using the SPF

First observation: we can represent $A(t)$ on a graph!

Lemma: There is always an optimal solution for the mouse with a disconnected union of singletons, pairs, and odd cycles.
A better bound using the SPF

First observation: we can represent $A(t)$ on a graph!

**Lemma:** There is always an optimal solution for the mouse with a disconnected union of singletons, pairs, and odd cycles.
A better bound using the SPF

First observation: we can represent $A(t)$ on a graph!

**Lemma:** There is always an optimal solution for the mouse with a disconnected union of singletons, pairs, and odd cycles.

- $n_1 \ll \text{singleton}$ (Here, $n_1 = 1$)
- A pair
- An odd cycle
First observation: we can represent $A(t)$ on a graph!

**Lemma:** There is always an optimal solution for the mouse with a disconnected union of singletons, pairs, and odd cycles.

$n_1 \ll \text{singleton} \quad (n_1=1 \text{ here})$

A pair

An odd cycle

**Lemma:** The SPF is equal to $2/(n+n_1)$ (when the optimal decomposition is known). In this case the dimension of the optimal primal solutions $P_t$ is the number of pairs.
The synchronizing function on practical examples

- Cerny’s automaton

\[ n_1 \ll \text{singleton} \] (\(n_1=1\) here)
The synchronizing function on practical examples

- Cerny’s automaton

If \( n - 2 \) singletons,

If \( n - 3 \) singletons,

If \( n - 4 \) singletons,

\( n_1 \leftarrow \text{singleton} \) (\( n_1 = 1 \) here)
First observation: we can represent $A(t)$ on a graph!

**Lemma:** There is always an optimal solution for the mouse with a disconnected union of singletons, pairs, and odd cycles.

$n_1 \ll \text{singleton} \quad (n_1=1 \text{ here})$

**Lemma:** The SPF is equal to $2/(n+n_1)$ (when the optimal decomposition is known). In this case the dimension of the optimal primal solutions $P_t$ is the number of pairs.
A better bound using the SPF

First observation: we can represent $A(t)$ on a graph!

Lemma: There is always an optimal solution for the mouse with a disconnected union of singletons, pairs, and odd cycles

Lemma: The SPF is equal to $\frac{2}{n+n_1}$ (when the optimal decomposition is known). In this case the dimension of the optimal primal solutions $P_t$ is the number of pairs.

Lemma: the dimension of the optimal primal Solutions has to decrease If $k(t)$ remains constant.
A better bound using the SPF

**Theorem:** in a synchronizing automaton with $n$ states,

$$T_3 \leq \sum_{s=0}^{n-1} \left(\left\lfloor (n - s)/2 \right\rfloor + 1 \right) = \frac{n(n + 4)}{4} - \frac{n \ mod(2)}{4}.$$

...
A better bound using the SPF

**Theorem:** in a synchronizing automaton with \( n \) states,

\[ T_3 \approx O(n^2/6.4\ldots) \]
Outline

- Synchronizing automata, Cerny’s conjecture, and previous approaches
- The synchronizing probability function and previous results
- New results:
  - a new upper bound (on a related quantity)
  - a counterexample
- Discussion
A conjecture

- Observation: At some fixed times, the value of the function is always higher (or equal) than Cerny’s automaton

Conjecture: It is always the case

For any synchronizing automaton and for any $j \geq 1$, $j \leq n - 1$,

$$k(1 + (j - 1)(n + 1)) \geq j/(n - 1).$$
Observation: At some fixed times, the value of the function is always higher (or equal) than Cerny’s automaton.

An easier conjecture on the triple rendezvous time:

For any synchronizing automaton, $T_3 \leq n + 2$. 
Counter example

Automaton with 9 states, $k(11)=2/9$ and $T_3=12 = n+3$

Contradicts both conjectures

An easier conjecture on the triple rendezvous time:

For any synchronizing automaton, $T_3 \leq n + 2$. 
Comparison of the SPF

Counterexample in black
Černý’s automaton with 9 states in dashed
Family extension

Extension of the family to 11 and 13 states

It can be extended to any odd number
Outline

• Synchronizing automata, Cerny’s conjecture, and previous approaches

• The synchronizing probability function and previous results

• New results: a counterexample and a new upper bound (on a related quantity)

• Discussion
Conclusion and future work

- **Future work:** Plenty of things!
  - What with **other automata:** non-synchronizing automata, Non-deterministic...
  - **Particular cases**
  - Improve the bound on $T_3 \rightarrow O(n)\ ? \quad n+3 < B < n^2/6.4$
  - Use our concepts to **generate slowly synchronizing automata**

- **Applications!**

- Our approach tried to **connect** this longstanding problem with other fields of mathematics.
  The connection seems to bear some sense and suggests new questions.
Questions?

More on:
http://perso.uclouvain.be/raphael.jungers