Higher spherical polynomials and holonomic systems

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Review on Gegenbauer polynomials

For integers $d > 2$, $\nu \geq 0$ and variable vectors $x = (x_i)_{1 \leq i \leq d}$ and $y = (y_i)_{1 \leq i \leq d}$, we consider polynomials $\tilde{P}_\nu(x, y)$ in $2d$ variables which satisfy the following three conditions.

(i) $\tilde{P}_\nu(xg, yg) = \tilde{P}_\nu(x, y)$ for any $g \in O(d)$.

(ii) $\tilde{P}_\nu(ax, by) = (ab)^\nu \tilde{P}_\nu(x, y)$ for any $a, b \in \mathbb{C}$, i.e. homogeneous of degree $\nu$ for each $x, y$.

(iii) $\tilde{P}_\nu(x, y)$ is harmonic for each $x$ and $y$.

By (i) and (ii), $\tilde{P}_\nu(x, y)$ is determined by $\tilde{P}_\nu(x, 1)$, in addition we have $\tilde{P}_\nu(xg_1, 1) = \tilde{P}_\nu(x, 1)$ for $g_1 \in O(d - 1)$. So $\tilde{P}(x, 1)$ is the spherical function of class one representation of $O(d)$. In this case, we can write $\tilde{P}_\nu(x, y) = P_\nu((x, y), n(x)n(y))$ by a polynomial $P_\nu(*, *)$ in two variables. The polynomial $P_\nu(t, 1)$ is the so-called Gegenbauer polynomial. When $d = 3$, this is the Legendre polynomials.
Interesting facts on Gegenbauer polynomials

We review what is known for Gegenbauer polynomials. We fix $d$.

(1) For each $\nu$, the polynomial $\tilde{P}_\nu$ or associated $P_\nu$ which satisfy (i), (ii), (iii) is determined uniquely up to constant.

(2) They satisfy the following generating function.

$$\frac{1}{(1 - 2tu + (rs)u^2)^{(d-2)/2}} = \sum_{\nu=0}^{\infty} P_\nu(t, rs)u^\nu,$$

where $r = (x, y)$, $r = n(x)$, $s = n(y)$, and $u$ is an indeterminant.

(3) The polynomials $P_\nu(t, 1)$ form a complete set of orthogonal polynomials with respect to the measure

$$\int_{-1}^{1} f(t)g(t)(1 - t^2)^{(d-3)/2} dt.$$
(4) The polynomial $P_\nu(t, 1)$ satisfies the following differential equation of Fuchsian type:

$$(1 - t^2) \frac{d^2 y}{dt^2} - (d - 1)t \frac{dy}{dt} + \nu(\nu + d - 2)y = 0.$$ 

(5) We have

$$P_\nu(t, 1) = \frac{(\nu + d - 3)!}{\nu!(d - 3)!} \, _2F_1 \left( \nu + d - 2, -\nu; \frac{d - 1}{2}; \frac{1 - t}{2} \right).$$

where $_2F_1$ is the hypergeometric function defined by

$$\, _2F_1(a, b; c; x) = \sum_{n=0}^{\infty} \frac{(a)_n(b)_n}{n!(c)_n} x^n$$

and $(\alpha)_n = \alpha(\alpha + 1) \cdots (\alpha + n - 1)$. 
For fixed $d$, $n$ with $d > 2n$, we consider $n \times d$ matrices $X = (x_{i\mu})$, $Y = (y_{i\mu})$ of variables. For each integer $\nu \geq 0$, we consider polynomials $\tilde{P}_\nu(X, Y)$ which satisfy the following three conditions.

1. $\tilde{P}_\nu(Xg, Yg) = \tilde{P}_\nu(X, Y)$ for all $g \in O(d)$.
2. $\tilde{P}_\nu(AX, BY) = \det(AB)^\nu \tilde{P}_\nu(X, Y)$ for any $A, B \in GL(n)$.
3. The polynomial $\tilde{P}_\nu$ is pluri-harmonic with respect to each $X$ and $Y$, that is,

$$\Delta_{ij}(X)\tilde{P}_\nu = \Delta_{ij}(Y)\tilde{P}_\nu = 0,$$

where

$$\Delta_{ij}(X) = \sum_{\nu=1}^{d} \frac{\partial^2}{\partial x_{i\mu} \partial x_{j\mu}}, \quad \Delta_{ij}(Y) = \sum_{\nu=1}^{d} \frac{\partial^2}{\partial y_{i\mu} \partial y_{j\mu}}.$$

By the condition (1), we have $\tilde{P}_\nu(X, Y) = P(X^t X, Y^t Y, X^t Y)$ for some polynomial $P(R, S, T)$, where $R, S, T$ are $n \times n$ matrices of variables and $R, S$ are symmetric.
Problems

Give the same theory for general $n$ as for Gegenbauer polynomials.

- Uniqueness of $P_\nu$ for each $\nu$ up to constant.
- Generating function for $n = 2$ and some explicit description for general $n$.
- Natural measure.
- Holonomic system that the radial part $Q_\nu$ of $P_\nu$ satisfies.
- Connection with Constantine-Muirhead generalized hypergeometric functions.


*Today we omit more general setting we have, containing works with Don Zagier.*
Uniqueness

The uniqueness of $P_\nu$ up to constant for each $\nu$ is proved by representation theory. We denote by $H(n, d)$ the space of pluri-harmonic polynomials $P(X)$ in components of $n \times d$ matrices. Then $GL(n) \times O(d)$ acts on $H(n, d)$ by $P(X) \rightarrow P(AXg)$. ($A \in GL(n)$, $g \in O(d)$). The decomposition of this representation on $H(n, d)$ is known by Kashiwara and Vergne and the multiplicity is one for each irreducible representation. We denote by $\rho_\nu$ the representation given by the space

$$\{P(X) \in H(n, d); P(AX) = \det(A)^\nu P(X)\}.$$ 

Our $\tilde{P}_\nu(X, Y)$ is in the representation space of the irreducible representation $\rho_\nu \otimes \rho_\nu$ of $(GL(n) \times O(d))^2$ and invariant by the action of $O(d) \subset O(d) \times O(d)$. This is unique up to constant by general theory.
Generating function for $n = 2$

When $n = 2$ and $d$ fixed, the polynomials $P_{\nu}(R, S, T)$ are given by the following generating function (up to constant):

$$
\frac{1}{R^{(d-5)/2} \sqrt{\Delta_0^2 - 4f_3 u^2}} = \sum_{\nu=0}^{\infty} P_{\nu}(R, S, T) u^\nu,
$$

where we put

$$
f_1 = \det(T), \quad f_2 = \det(RS), \quad f_3 = \det \begin{pmatrix} R & T \\ tT & S \end{pmatrix}
$$

and

$$
\Delta_0 = 1 - 2f_1 u + f_2 u^2
$$

$$
R = (\Delta_0 + \sqrt{\Delta_0^2 - 4f_3 u^2})/2.
$$

We have no results for $n \geq 3$ on generating functions.
A formula for $P_{\nu}$ for general $n$

For general $n$, we assume that $d > 2n$. For $n \times d$ matrices $X = (x_{i\mu})$, $Y = (y_{i\mu})$ of variables, we write

$$
\Delta_{ij}(X, Y) = \sum_{\mu=1}^{d} \frac{\partial^2}{\partial x_{i\mu} \partial y_{j\mu}}.
$$

By $R = X^t X$, $S = Y^t Y$ and $T = X^t Y$, we can rewrite $\Delta_{ij}(X, Y)$ by derivations with respect to $r_{ij}$, $s_{ij}$ and $t_{ij}$ at most of order two.

**Theorem**

For each $\nu \geq 0$, our polynomial $P_{\nu}(R, S, T)$ is given by the following formula up to constant:

$$
P_{\nu}(R, S, T) = \det(RS)^{(d-n+1)/2+\nu-1} \det(\Delta_{ij}(X, Y))^\nu \det(RS)^{(n+1-d)/2}
$$

Keypoint: To prove RHS is a polynomial (joint with Y. Hyogo.)
An invariant measure of the action of $GL(n) \times O(d)$ on the space $H(n, d)$ of pluri-harmonic polynomials in $X \in M_{n,d}(\mathbb{R})$ is given by

$$\int_{M_{n,d}(\mathbb{R})} e^{-Tr(X^tX)} F(X) dX.$$ 

Since $\widetilde{P}_\nu(X, Y) \in H(n, d) \otimes H(n, d)$, we take the double integral of the above and by rewriting we get the natural inner product in our case. This is given for $F(R, S, T)$ and $G(R, S, T)$ by

$$(F, G) = \int_{S_{2n}} F^\top G \det \begin{pmatrix} R & T \\ t^T & S \end{pmatrix}^{(d-2n-1)/2} dR dS dT,$$

where $S_{2n}$ is the set of $2n \times 2n$ positive definite matrices. For any $\nu \neq \mu$, we have $(P_\nu(R, S, T), P_\mu(R, S, T)) = 0$. 
For our $P_\nu(R, S, T)$, we have

$$P_\nu(R, S, T) = \det(RS)^{\nu/2} P_\nu(1_n, 1_n, R^{-1/2} TS^{-1/2})$$

$$= \det(RS)^{\nu/2} P_\nu(1_n, 1_n, P_1(R^{-1/2} TS^{-1/2}) P_2)$$

for any orthogonal polynomials $P_1, P_2$. So denoting by $\lambda_i$ the spectres of $R^{-1/2} TS^{-1/2}$ in $O(d) \setminus M_n(\mathbb{R}) / O(d)$, we have

$$P_\nu(R, S, T) = \det(RS)^{\nu/2} Q_\nu(\lambda_1, \ldots, \lambda_n)$$

for some polynomial $Q_\nu$ in $\lambda_1, \ldots, \lambda_n$. (Actually $Q_\nu$ for even $\nu$ or $Q_\nu / \prod_{i=1}^n \lambda_i$ for odd $\nu$ is a symmetric polynomial in $\lambda_1^2, \ldots, \lambda_n^2$. )
We can interpret the condition of pluri-harmonicity as a system of $n$ differential operators on the radial part. We define

$$\mathbb{D}_k = (1 - \lambda_k^2) \frac{\partial^2}{\partial \lambda_k^2} + \left( -(d - 2n + 1) \lambda_k + \sum_{l \neq k} \frac{\lambda_k(1 - \lambda_k^2)}{\lambda_k^2 - \lambda_l^2} \right) \frac{\partial}{\partial \lambda_k}$$

$$+ \sum_{l \neq k} \frac{(1 - \lambda_l^2) \lambda_l}{(\lambda_l^2 - \lambda_k^2)} \frac{\partial}{\partial \lambda_l} + \nu(\nu + d - n - 1) \quad (1 \leq k \leq n).$$

and regard this as the generalized Gegenbauer differential equation:

**Theorem**

The polynomial $Q_\nu(\lambda_1, \ldots, \lambda_n)$ is a polynomial solution of the system

$$\mathbb{D}_k Q = 0 \quad (1 \leq k \leq n).$$
Holonomic system

**Theorem**

*For any complex parameters \( d \) and \( \nu \), the system of differential equations*

\[
\mathbb{D}_k Q = 0 \quad (1 \leq k \leq n)
\]

*is a holonomic system of rank \( 2^n \).*

We do not review the definition of holonomic system here, but this theorem includes the following claim.

At any points \((\lambda_1, \ldots, \lambda_n) \in \mathbb{C}^n\) outside the singular locus \( \lambda_k \neq \pm \lambda_l \) and \( \lambda_k = \pm 1 \), the linear space of solutions of this system is exactly \( 2^n \)-dimensional.

This is interesting since examples of explicitly written holonomic system is rather rare.
Constantine hypergeometric series

For any $a \in \mathbb{C}$ and for any partition $\kappa = (k_1, \ldots, k_n)$ of $k \in \mathbb{Z}_{\geq 0}$ with $k_1 \geq k_2 \geq \cdots \geq k_n \geq 0$ and $k = k_1 + \cdots + k_n$, we write

$$(a)_\kappa = \prod_{i=1}^{n} (a - (i - 1)/2)_{k_i}.$$ 

We denote by $C_\kappa(z)$, $(z = (z_1, \ldots, z_n))$, the zonal spherical polynomial on $GL(n)/O(n)$ for the partition parameter $\kappa$. Then for complex parameters $a, b, c$, we define generalized hypergeometric function by

$$2F_1(a, b, c; z) = \sum_\kappa \frac{(a)_\kappa (b)_\kappa}{(c)_\kappa} \frac{C_\kappa(z)}{(k_1 + \cdots + k_n)!}.$$ 

where $\kappa$ runs over the partitions of any $k \geq 0$ at most into $n$ parts.
Muirhead theorem on differential equations

We define differential operators \( \tilde{D}_i(a, b, c) = \tilde{D}_i \) in \((z_1, \ldots, z_n)\) by

\[
\tilde{D}_i = z_i (1 - z_i) \frac{\partial^2}{\partial z_i^2} + \left( c - \frac{n - 1}{2} - (a + b + 1 - \frac{n - 1}{2})z_i \right) \frac{\partial}{\partial z_i} + \frac{1}{2} \sum_{j \neq i} \frac{z_i (1 - z_i)}{z_i - z_j} \frac{\partial}{\partial z_j} - \frac{1}{2} \sum_{j \neq i} \frac{z_j (1 - z_j)}{z_i - z_j} \frac{\partial}{\partial z_j}.
\]

**Theorem (Muirhead)**

The series \( _2F_1(a, b; c; z) \) is the unique solution of the system

\[
\tilde{D}_i(a, b, c)f = abf \quad (1 \leq i \leq n)
\]

such that

1. \( f \) is holomorphic near \( z = 0 \) and \( f(0, \ldots, 0) = 1 \),
2. \( f(z_1, \ldots, z_n) \) is symmetric with respect to the variables \( z_i \).
Relation to our polynomials

Let $d$, $n$, $\nu$ be positive integers. If we change $z_i = \lambda_i^2$, $a = -\nu/2$, $b = (\nu + d - n - 1)/2$, $c = n/2$, then we have

$$D_i = 4(\tilde{D}_i(a, b, c) - ab).$$

We denote by $Q_\nu(\lambda_1, \ldots, \lambda_n)$ our polynomials as before. We have

**Theorem**

(1) If $\nu$ is even, then we have

$$Q_\nu(\lambda_1, \ldots, \lambda_n) = 2F_1\left(-\frac{\nu}{2}, \frac{1}{2}(\nu + d - n - 1); \frac{n}{2}; (\lambda_1^2, \ldots, \lambda_n^2)\right).$$

(2) If $\nu$ is odd, then we have

$$\frac{Q_\nu(\lambda_1, \ldots, \lambda_n)}{\lambda_1 \cdots \lambda_n} = 2F_1\left(-\frac{\nu - 1}{2}, \frac{\nu + d - n}{2}; \frac{n}{2} + 1; (\lambda_1^2, \ldots, \lambda_n^2)\right).$$
Spherical functions on Grassmannian manifolds

We assume \( d > 2n \). The oriented Grassmann manifold consisting of \( n \) dim. oriented subspaces in the fixed \( d \)-dim. real vector space is identified with \( G_{d,n}^0 = (SO(n) \times O(d - n)) \backslash O(d) \). For each irreducible subrepresentation of \( O(d) \) on \( L^2(G_{d,n}^0) \), we have the unique function up to constant which is bi-\( SO(n) \times O(d - n) \) invariant. This is called zonal spherical. We have

\[
L^2(G_{d,n}^0)_{SO(n) \times O(d-n)} \cong L^2(G_{d,n}^0 \times G_{d,n}^0)^{O(d)}.
\]

If we write \( \widetilde{P}_\nu(X, Y) = P_\nu(X^t X, Y^t Y, X^t Y) \) and take a natural restriction of a function \( \widetilde{P}_\nu(X, Y) \) on \( M_{d,n} \times M_{d,n} \) to \( G_{d,n}^0 \times G_{d,n}^0 \), then this gives the zonal spherical function corresponding to the irreducible representation of \( O(d) \) with Young diagram parameter \((\nu, \ldots, \nu, 0, \ldots, 0; (-1)^{n\nu})\).
Motivation

We consider two bounded symmetric domains: $\Delta \subset D$, and biholomorphic automorphism groups: $Aut(\Delta) \subset Aut(D)$.

$V$: finite dim. vector space over $\mathbb{C}$. Automorphy factor $J_\Delta$ in $GL(V)$ and $J_\Delta$ in $GL(1)$, i.e. $F|_{J\Delta}[g] = J_\Delta(g, Z)^{-1}F(gZ)$ (for $F : \Delta \to V$) is an action. Let $\mathbb{D}$ be a linear $V$-valued differential operator with constant coefficients satisfying

**Condition on $\mathbb{D}$**

\[
\begin{array}{ccc}
Hol(D, \mathbb{C}) & \xrightarrow{\mathbb{D}} & Hol(D, V) \\
\downarrow |_{J_D}[g] & & \downarrow |_{J_\Delta}[g] \\
Hol(D, \mathbb{C}) & \xrightarrow{\text{Res.}} & Hol(\Delta, V)
\end{array}
\]

$g \in Aut(\Delta) \subset Aut(D)$.

**Characterization:** We have $\mathbb{D} = P\left(\frac{\partial}{\partial z_1}, \ldots, \frac{\partial}{\partial z_n}\right)$ ($Z = (z_i) \in D$) with invariant harmonic polynomial $P$ of similar sort.
Why such $\mathcal{D}$ is important?

We denote by $H_n$ the Siegel upper half space of degree $n$:

$$H_n = \{ Z = X + iY \in M_n(\mathbb{C}); Z = \begin{bmatrix} X \\ Y \end{bmatrix}, \text{Im}(Z) > 0 \}.$$ 

The domains $\Delta = H_n \times H_n \subset H_{2n}$ and the automorphy factor $\det^k$ and the target $\det^{k+\nu}$ are the cases we treated today and our $P_\nu$ gives the above $\mathcal{D}$. Such $\mathcal{D}$ is important since

1. We can calculate the special values of standard $L$ functions of automorphic forms at various critical points by using $\mathcal{D}$.

2. Starting from given automorphic forms, by using $\mathcal{D}$, we can construct new automorphic forms, which are often difficult to construct by any other methods.

3. The differential operators $\mathcal{D}$ are characterized by certain polynomials and these polynomials are sources of new special functions.

In this talk, we emphasized the viewpoint (3) above.