Tilings of the Sphere by Congruent Pentagons

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Introduction

Tilings of the sphere by congruent triangles.

- Sommerville[1923]: partial classification, isosceles triangles.
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Tilings of the sphere by congruent triangles.

- Sommerville[1923]: partial classification, isosceles triangles.
- Ueno-Agaoka[2002]: complete classification, about 20 families.

\[ \alpha + \beta + \gamma = 2\pi \]
Introduction

Why sphere and pentagon?

- Finitely many tiles, easier than plane tilings.
- Tiling of sphere by $n$-gon $\iff n = 3, 4, 5$. 5 is the "other extreme".
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(Geometrical) congruence = Edge + Angle congruences.
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- Tiling of sphere by $n$-gon $\implies n = 3, 4, 5$.  
  5 is the “other extreme”.

(Geometrical) congruence $= \text{Edge} + \text{Angle congruences}$.  

However, for $n \geq 4$, we need to separately consider edge and angle congruences.
Introduction

combinatorial congruence

edge congruence

angle congruence

geometrical congruence
Pentagonal Subdivision

Pentagonal subdivision:
- Start with any tiling of oriented surface by polygons.
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- Start with any tiling of oriented surface by polygons.
- Each $n$-gon is divided into $n$ pentagons.
Pentagonal Subdivision

Pentagonal subdivision of platonic solids
⇒ Geometrically congruent tiling of the sphere.

\[ \begin{align*}
\alpha + \beta + \gamma &= 2\pi, \\
\delta &= \frac{2\pi}{3}, \\
\epsilon &= \frac{2\pi}{4}.
\end{align*} \]
Pentagonal Subdivision

Pentagonal subdivision of platonic solids
⇒ Geometrically congruent tiling of the sphere.

\[ \alpha + \beta + \gamma = 2\pi, \quad \delta = \frac{2\pi}{3}, \quad \epsilon = \frac{2\pi}{4} \]

two degrees of freedom
Pentagonal Subdivision

Pentagonal subdivision of platonic solids

$\implies$ Geometrically congruent tiling of the sphere.

two pairs of equal edges

$\alpha + \beta + \gamma = 2\pi$, $\delta = \frac{2}{3}\pi$, $\epsilon = \frac{2}{4}\pi$

two degrees of freedom
Pentagonal Subdivision

Pentagonal subdivision of

- tetrahedron: \( f = 12 \)
- (cube, octahedron): \( f = 24 \)
- (dodecahedron, icosahedron): \( f = 60 \)
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Are these the only tilings of the sphere by geometrically congruent pentagons?

**Theorem** [Gao-Shi-Y, 2013]
Yes for $f = 12$. 
Pentagonal Subdivision

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Are these the only tilings of the sphere by geometrically congruent pentagons?

**Theorem** [Gao-Shi-Y, 2013]
Yes for $f = 12$.

Will always assume $f > 12$. 
Neighborhood

Numerical:

- $v - e + f = 2$.
- $2e = 5f$.
- $3v_3 + 4v_4 + 5v_5 + \cdots = 5f$. ($v_i =$ number of degree $i$ vertices.)
- $v_3 + v_4 + v_5 + \cdots = v$. 

Consequences:

- $v_3 = 20 + 2v_4 + 5v_5 + 8v_6 + \cdots$: degree 3 vertices dominate.
- $12 - 6 = v_4 + 2v_5 + 3v_6 + \cdots$: $f$ is even, $f \geq 12$.
- There is a tile with four degree 3 vertices, and the 5-th vertex has degree 3, 4, or 5.
Neighborhood

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Consequences:

- $v_3 = 20 + 2v_4 + 5v_5 + 8v_6 + \cdots$: degree 3 vertices dominate.
- $\frac{1}{2}f - 6 = v_4 + 2v_5 + 3v_6 + \cdots$: $f$ is even, $f \geq 12$.
- There is a tile with four deg 3 vertices, and the 5-th vertex has degree 3, 4, or 5.
Neighborhood

Assume there is a tile with all vertices having degree 3.
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\[ \Rightarrow \text{5 edge combos.} \]
Neighborhood

Assume there is a tile with all vertices having degree 3.

⇒ 5 edge combos. We will study three cases.
Neighborshood

Extend to neighborhood.

\[ a^3 b^2 \hspace{1cm} a^3 b c \hspace{1cm} a^2 b^2 c \]
Neighborhood

Extend to neighbourhood.

\[
\begin{align*}
& a^3 b^2 \\
& a^3 b c \\
& a^2 b^2 c
\end{align*}
\]

edge congruent nd.

\[
\begin{align*}
& \\
\end{align*}
\]
Neighborhood

Extend to neighbourhood.

Then add consistent angles \( \implies f = 12 \).
Neighborhood

Extend to neighbourhood.

Then add consistent angles $\Rightarrow f = 12$
Neighborhood

Extend to neighbourhood.

\[
\begin{align*}
& a^3 b^2 \\
& a^3 b c \\
& a^2 b^2 c
\end{align*}
\]
Neighborhood

Extend to neighbourhood.

\[
\begin{align*}
&\text{I} & a^3 b^2 \\
&\text{II} & a^3 b c \\
&\text{III} & a^2 b^2 c
\end{align*}
\]
Neighborhood

Extend to neighbourhood.

\[
\begin{align*}
\theta_1 & = a^3 b^2 \theta_2 \\
\phi_1 & = a^3 b c \\
\phi_2 & = a^2 b^2 c
\end{align*}
\]

\[
\begin{align*}
\alpha & = 12 \\
\theta_1 & = 12 \\
\phi_1 & = 12 \\
\phi_2 & = 12
\end{align*}
\]

1 tiling

3 tilings
Neighborhood

Four “geometrically congruent” neighbourhood tilings.

Compute the angles.
Four “geometrically congruent” neighbourhood tilings.

Compute the angles.

<table>
<thead>
<tr>
<th>tiling</th>
<th>$\alpha$</th>
<th>$\theta_1$</th>
<th>$\theta_2$</th>
<th>$\phi_1$</th>
<th>$\phi_2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>II</td>
<td>$\frac{2}{3} \pi$</td>
<td>$\theta_1 + \theta_2 = \left( \frac{2}{3} + \frac{8}{f} \right) \pi$</td>
<td>$\left( \frac{1}{3} + \frac{4}{f} \right) \pi$</td>
<td>$\left( \frac{4}{3} - \frac{8}{f} \right) \pi$</td>
<td></td>
</tr>
<tr>
<td>III$_1$</td>
<td>$\frac{2}{3} \pi$</td>
<td>$\left( \frac{1}{3} + \frac{4}{f} \right) \pi$</td>
<td>$\left( \frac{4}{3} - \frac{8}{f} \right) \pi$</td>
<td>$\left( \frac{2}{3} + \frac{16}{f} \right) \pi$</td>
<td></td>
</tr>
<tr>
<td>III$_2$</td>
<td>$\frac{2}{3} \pi$</td>
<td>$\left( \frac{5}{6} - \frac{2}{f} \right) \pi$</td>
<td>$\left( -\frac{1}{6} + \frac{10}{f} \right) \pi$</td>
<td>$\left( \frac{1}{3} + \frac{4}{f} \right) \pi$</td>
<td>$\left( \frac{4}{3} - \frac{8}{f} \right) \pi$</td>
</tr>
<tr>
<td>III$_3$</td>
<td>$\frac{2}{3} \pi$</td>
<td>$\left( -\frac{1}{6} + \frac{10}{f} \right) \pi$</td>
<td>$\left( \frac{5}{6} - \frac{2}{f} \right) \pi$</td>
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</tr>
</tbody>
</table>
Tiling

Two further tools for constructing the whole tiling:

- All possible angle combinations.
- Existence of spherical pentagon.
Tiling

Consider angle combination.
Tiling

Consider angle combination.

For \( \text{III}_1 \)

\[
\phi_2 = \left( -\frac{2}{3} + \frac{16}{f} \right) \pi > 0 \implies f < 24.
\]
Tiling

Consider angle combination.

For \( \text{III}_1 \)

\[
\phi_2 = \left(-\frac{2}{3} + \frac{16}{f}\right) \pi > 0 \implies f < 24.
\]

For \( f = 18 \)

\[
\alpha = \frac{2}{3} \pi, \quad \theta_1 = \frac{5}{9} \pi, \quad \theta_2 = \frac{8}{9} \pi, \quad \phi_1 = \frac{8}{9} \pi, \quad \phi_2 = \frac{2}{9} \pi.
\]

This implies \( \theta_1 \theta_2 \cdots = \theta_1^2 \theta_2. \)
Tiling

Consider angle combination.

For $\text{III}_1$

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For $f = 18$

$$\alpha = \frac{2}{3} \pi, \quad \theta_1 = \frac{5}{9} \pi, \quad \theta_2 = \frac{8}{9} \pi, \quad \phi_1 = \frac{8}{9} \pi, \quad \phi_2 = \frac{2}{9} \pi.$$ 

This implies $\theta_1 \theta_2 \cdots = \theta_1^2 \theta_2$. 

\begin{center}
\begin{tikzpicture}
\begin{scope}[scale=0.5]
\node at (0,0) {$\alpha$};
\node at (0,1) {$\theta_1$};
\node at (1,2) {$\phi_2$};
\node at (2,1) {$\theta_2$};
\node at (1,0) {$\phi_1$};
\node at (0,-1) {$\theta_1$};
\node at (-1,0) {$\theta_2$};
\node at (-2,-1) {$\phi_2$};
\node at (-1,-2) {$\phi_1$};
\node at (0,-3) {$\alpha$};
\end{scope}
\end{tikzpicture}
\end{center}
Tiling

Consider angle combination.

For III$_1$

$$\phi_2 = \left(-\frac{2}{3} + \frac{16}{f}\right)\pi > 0 \implies f < 24.$$ 

For $f = 18$

$$\alpha = \frac{2}{3}\pi, \quad \theta_1 = \frac{5}{9}\pi, \quad \theta_2 = \frac{8}{9}\pi, \quad \phi_1 = \frac{8}{9}\pi, \quad \phi_2 = \frac{2}{9}\pi.$$ 

This implies $\theta_1\theta_2\cdots = \theta_1^2\theta_2$. But $\theta_1$ is $a^2$-vertex, contradiction.
Consider **existence of spherical pentagon**.

**Lemma**
If the boundary of a spherical pentagon is simple, and two pairs of edges are equal, then \( \beta > \gamma \iff \delta < \epsilon \).

Without Lemma, we get 7 type III tilings.
Tiling

Consider existence of spherical pentagon.
Tiling

Consider existence of spherical pentagon.

For $\text{III}_2$

$$\theta_2 = \left(-\frac{1}{6} + \frac{10}{f}\right) \pi > 0 \implies f < 60.$$
Tiling

Consider existence of spherical pentagon.

For $\text{III}_2$

$$\theta_2 = \left(-\frac{1}{6} + \frac{10}{f}\right)\pi > 0 \implies f < 60.$$ 

Alternative argument reduces to two possible $f$

- $f = 24$: $\alpha^3, \theta_1\theta_2\phi_2, \theta_1^2\phi_1, \phi_1^2\phi_2, \theta_1^2\theta_2^2, \theta_1\theta_2\phi_1^2, \theta_2^2\phi_1\phi_2, \phi_1^4.$

- $f = 36$: $\alpha^3, \theta_1\theta_2\phi_2, \theta_1^2\phi_1, \phi_1^2\phi_2, \alpha\theta_1\theta_2\phi_1, \alpha\theta_2^2\phi_2.$
Tiling

Consider existence of spherical pentagon.

For $\text{III}_2$

$$\theta_2 = (-\frac{1}{6} + \frac{10}{f}) \pi > 0 \implies f < 60.$$ 

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- $f = 24$: $\alpha^3, \theta_1\theta_2\phi_2, \theta_1^2\phi_1, \phi_1^2\phi_2, \theta_1^2\theta_2^2, \theta_1\theta_2\phi_1^2, \theta_2^2\phi_1\phi_2, \phi_1^4$.
- $f = 36$: $\alpha^3, \theta_1\theta_2\phi_2, \theta_1^2\phi_1, \phi_1^2\phi_2, \alpha\theta_1\theta_2\phi_1, \alpha\theta_2^2\phi_2$.

$f = 36 \implies$ contradiction, similar to $\text{III}_1$.

$f = 24 \implies$

$$\alpha = \frac{2}{3}\pi, \quad \theta_1 = \frac{3}{4}\pi, \quad \theta_2 = \frac{1}{4}\pi, \quad \phi_1 = \frac{1}{2}\pi, \quad \phi_2 = \pi.$$
Tiling

For $\text{III}_2$, $f = 24$ gives

\[ \alpha = \frac{2}{3}\pi, \quad \theta_1 = \frac{3}{4}\pi, \quad \theta_2 = \frac{1}{4}\pi, \quad \phi_1 = \frac{1}{2}\pi, \quad \phi_2 = \pi. \]
For $\text{III}_2$, $f = 24$ gives

$$\alpha = \frac{2}{3}\pi, \quad \theta_1 = \frac{3}{4}\pi, \quad \theta_2 = \frac{1}{4}\pi, \quad \phi_1 = \frac{1}{2}\pi, \quad \phi_2 = \pi.$$
Tiling

For $\text{III}_2$, $f = 24$ gives

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A general pentagon allows 7 free variables.

$a^3 b^2$ means 3 equations, and allows $7 - 3 = 4$ free variables. So there is 1 relation among 5 angles.
Tiling

The five angles are contradictory!!!

octdrant
Future

Assume there is a tile with all vertices having degree 3, and $f > 12$. 
Assume there is a tile with all vertices having degree 3, and $f > 12$. 

\[ a^5 \quad a^4b \] 

Remaining:

- There is a tile with four deg 3 vertices, and the fifth vertex has degree 4 or 5.
- Edge combo $a^5$.
- Edge combo $a^4b$, most difficult.
Assume there is a tile with all vertices having degree 3, and $f > 12$.

Remaining:
- There is a tile with four deg 3 vertices, and the fifth vertex has deg 4 or 5.
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Future

Edge combo $a^5$ allows 3 free variables.
Angle combinations at deg 3 vertices mostly give 3 equations.
The pentagon can be mostly determined.

<table>
<thead>
<tr>
<th>Necessary</th>
<th>Optional</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\alpha^3$</td>
<td></td>
</tr>
<tr>
<td>$\alpha \beta^2$</td>
<td></td>
</tr>
<tr>
<td>$\alpha \beta \gamma$</td>
<td>$\alpha^3$</td>
</tr>
<tr>
<td>$\alpha \beta^2$</td>
<td>$\alpha^2 \gamma$ $\gamma^3$</td>
</tr>
</tbody>
</table>

\begin{align*}
\alpha \beta \gamma & : \alpha^2 \delta, \beta^2 \delta, \beta^3 \\
\alpha \beta \gamma & : \beta^2 \delta, \beta^3 \\
\alpha \beta \gamma & : \delta^3 \\
\alpha \beta^2 & : \gamma^2 \delta, \beta^2 \epsilon \\
\alpha \beta^2 & : \alpha^2 \gamma, \delta^3 \\
\end{align*}

\begin{align*}
\alpha \beta \gamma & : \beta^2 \delta, \beta^2 \epsilon \\
\alpha \beta \gamma & : \beta^2 \delta, \gamma^2 \epsilon, \alpha^3 \\
\alpha \beta \gamma & : \beta^2, \gamma^2 \epsilon \\
\alpha \beta \gamma & : \beta^2, \gamma^3 \\
\alpha \beta \gamma & : \beta^2, \epsilon^3 \\
\alpha \beta \gamma & : \alpha^2 \delta \\
\alpha^2 \epsilon & : \beta^2 \delta \\
\alpha^2 \epsilon & : \beta^3 \\
\beta \epsilon^2 & : \alpha^2 \epsilon \\
\beta \epsilon^2 & : \gamma^2 \delta \\
\beta \epsilon^2 & : \gamma^3 \\
\beta \epsilon^2 & : \epsilon^3 \\
\beta^2 \epsilon & : \beta^2 \epsilon \\
\beta^2 \epsilon & : \beta^3 \\
\beta^2 \epsilon & : \gamma^2 \delta \\
\beta^2 \epsilon & : \gamma^3 \\
\beta^2 \epsilon & : \epsilon^3 \\
\beta^2 \delta & : \delta \epsilon^2 \\
\beta^2 \delta & : \alpha^3 \\
\beta^2 \delta & : \beta^2 \epsilon \\
\beta^2 \delta & : \beta^3 \\
\beta^2 \delta & : \gamma^2 \delta \\
\beta^2 \delta & : \gamma^3 \\
\beta^2 \delta & : \epsilon^3 \\
\beta^2 \delta & : \alpha^3 \\
\end{align*}
Thank You