Coloring Invariants of Knots

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Knot theory and Fox $n$-coloring

- Knot: an embedding of a circle in 3-dimensional Euclidean space $R^3$ or $S^3$
- Two knots are equivalent if one can be transformed into the other via an ambient isotopy
- Here comes the question:

How to distinguish one knot from another?

unknot  trefoil knot  figure-eight knot
Theorem (Kurt Reidemeister 1927)

Two knot diagrams belonging to the same knot if and only if one can be obtained from the other by a sequence of three kinds of moves.

- A knot invariant is a “quantity” that is the same for equivalent knots.
- A “quantity” is a knot invariant if it is preserved under three Reidemeisters.
- For example: unknotting number, crossing number, genus, signature, knot group, Alexander polynomial, Jones polynomial, Vassiliev invariants, knot Floer homology, Khovanov homology...
Knot theory and Fox $n$-coloring

- An elementary invariant: Tricolorability
- Coloring each strand of the knot diagram with one of three colors (red, blue, green), such that at each crossing, the three incident strands are either all the same color or all different colors.

Theorem

Given a knot diagram $D$, the number of proper colorings $\text{Col}_3(D)$ is preserved under Reidemeister moves, hence is a knot invariant.
Knot theory and Fox $n$-coloring

Some properties of $Col_3(K)$:

- Each proper coloring corresponds to a representation from the knot group to the dihedral group of order 6
- $Col_3(K_1 \# K_2) = \frac{1}{3} Col_3(K_1) Col_3(K_2)$
- $Col_3(K)$ is always a power of 3, i.e. $Col_3(K) = 3^m$
- (Przytycki) $u(K) \geq \log_3(\text{Col}_3(K)) - 1$
A **quandle** (Joyce, Matveev 1982) $Q$ is a finite set with a binary operation $*: Q \times Q \rightarrow Q$, which satisfies

1. $a * a = a$ for any $a \in Q$
2. $x * a = b$ have the only solution $x \in Q$, for any $a, b \in Q$
3. $(a * b) * c = (a * c) * (b * c)$ for any $a, b, c \in Q$

For instance,

- **Trivial quandle**: $Q = \{a_1, \cdots, a_n\}$, define $a_i * a_j = a_i$.
- **Dihedral quandle**: $Q = \{0, 1, \cdots, n-1\}$, define $a * b = 2b - a \pmod{n}$.
- **Alexander quandle**: $Q$ is a $\mathbb{Z}[t, t^{-1}]$-module and $a * b = ta + (1 - t)b$. 
All elements of $Q$ are called colors. Given a diagram $D$ of a knot, we can sign each arc of $D$ with a color of $Q$. By a proper coloring of $D$ we mean that $D$ is colored in such a way that for each crossing of $D$, the relation $a \ast b = c$ holds.

\[
\begin{align*}
  \quad & \quad a \\
\downarrow & & \quad c = a \ast b \\
  b & \quad & \quad b
\end{align*}
\]

**Theorem**

*For a given quandle $Q$, the number of proper colorings is a knot invariant.*

- The dihedral quandle $D_3$ corresponds to the invariant $Col_3(K)$
- In general the dihedral quandle $D_n$ corresponds to the Fox $n$-coloring
Quandle and quandle homology

Proof:

\[ a \xrightarrow{a^*a} \]

\[ a \xrightarrow{a} \xrightarrow{b} a^*b \]

\[ a^*b \]

\[ a \xrightarrow{b} b^*c \]

\[ (a^*c)^* (b^*c) \]

\[ (a^*b)^*c \]
**Question**

*How to generalize the coloring invariant for a given quandle?*

1. For a given quandle, the coloring invariant is the number of proper colorings.

2. For a fixed colored knot diagram, if one can define a “colored knot invariant” then the set of all “colored knot invariants” is a generalized coloring invariant.

3. One of the easiest methods is counting the contribution of each crossing point.
Quandle and quandle homology

Quandle homology (Carter, Jelsovsky, Kamada, Langford, and Saito 1999)

- For a rack $X$ (a set satisfying quandle condition 2 and 3), let $C^R_n(X)$ be the free abelian group generated by $n$-tuples $(x_1, \cdots, x_n)$ of elements of $X$

- Define a homomorphism
  $$\partial_n(x_1, \cdots, x_n) = \sum_{i=1}^{n} (-1)^i [(x_1, \cdots, \overline{x_i}, \cdots, x_n) - (x_1 * x_i, \cdots, x_{i-1} * x_i, x_{i+1}, \cdots, x_n)]$$

- Let $C^D_n(X)$ be a subset of $C^R_n(X)$ generated by $n$-tuples $(x_1, \cdots, x_n)$ with $x_i = x_{i+1}$ for some $i \in \{1, \cdots, n-1\}$
Quandle and quandle homology

- \{ C^D_n(X), \partial_n \} is a subcomplex of \{ C^R_n(X), \partial_n \}

- Define \( C^Q_*(X) \) to be the quotient complex \( C^R_*(X)/C^D_*(X) \)

- Consider the homology groups and cohomology groups
  1. \( H^R_n(X; G) = H_n(C^R_*(X) \otimes G), H^R_n(X; G) = H^n(\text{Hom}(C^R_*(X) \otimes G)) \)
  2. \( H^D_n(X; G) = H_n(C^D_*(X) \otimes G), H^D_n(X; G) = H^n(\text{Hom}(C^D_*(X) \otimes G)) \)
  3. \( H^Q_n(X; G) = H_n(C^Q_*(X) \otimes G), H^Q_n(X; G) = H^n(\text{Hom}(C^Q_*(X) \otimes G)) \)

Theorem (Litherland and Nelson 2003)

There is a short exact sequence

\[ 0 \to H^D_n(X; G) \to H^R_n(X; G) \to H^Q_n(X; G) \to 0 \]
Applications of quandle homology and quandle cohomology:

- Quandle homology and quandle cohomology are invariants of quandle, i.e. they can be used to distinguish different quandles.
- Quandle 2-cocycle can be used to define a stronger coloring invariant of knots in $S^3$.
- Quandle 3-cocycle can be used to define a stronger coloring invariant of 2-knots in $S^4$.

Remarks: Most applications of cocycle invariants appear in 2-knot theory.
Let $X$ and $G$ be a finite quandle and an abelian group respectively, let \( \phi \in H^2_Q(X; G) \) be a quandle 2-cocycle, and $D$ a knot diagram.

- A proper coloring $\rho : Q(D) \to X$
- For a crossing point $c$, consider the contribution of $c$
  \[
  W_\phi(c, \rho) = \phi(\rho(x_i), \rho(x_j))\epsilon(c)
  \]
  Here $\epsilon(c)$ denotes the sign of $c$

- Consider the element of the group ring $\mathbb{Z}G$
  \[
  \Phi_\phi(D) = \sum_\rho \prod_c W_\phi(c, \rho)
  \]
Quandle and quandle homology

Theorem (Carter, Jelsovsky, Kamada, Langford, and Saito 2003)

- $\Phi_{\phi}(D)$ is invariant under Reidemeister moves, hence it defines an invariant of knots and links
- If $\phi_1$ and $\phi_2 \in Z^2_Q(X; G)$ are a pair of cohomologous cocycles, then $\Phi_{\phi_1}(D) = \Phi_{\phi_2}(D)$
- In particular if $\phi$ is a coboundary then $\Phi_{\phi}(D)$ is equal to the number of proper colorings

The key of the proof: 2-cocycle condition corresponds to the third Reidemeister move
Quandle and quandle homology

For each colored knot diagram, the 2-cocycle invariant concerns the contribution of each crossing, and then taking the sum of them.

**Question**

*Is it possible to find some other colored knot invariants, such that one can obtain a new state-sum invariant which can be regarded as a generalization of the number of proper colorings?*

Ongoing work: assume $G$ is non-abelian...... no progress at present : ( 
Recall that $\text{Col}_n(K)$ denotes the number of proper Fox $n$-colorings.

(Kauffman and Lopes 2008) Define $\text{min } \text{Col}_n(K)$ to be the minimum number of distinct colors that are needed to produce a non-trivial Fox $n$-coloring among all diagrams of $K$.

In general, min $\text{Col}_n(K)$ is very difficult to calculate.
Kauffman-Harary conjecture and its generalization

Conjecture (Kauffman and Harary 1999)

The minimum number of colors \( \min \text{Col}_p(D) \) of a reduced alternating knot diagram \( D \) with prime determinate \( p \) is exactly the crossing number of \( D \).

- (Marta M. Asaeda, Jozef H. Przytycki, Adam S. Sikora 2004) Kauffman-Harary conjecture holds for Montesinos knots
- (Thomas W. Mattman, Pablo Solis 2009) Kauffman-Harary conjecture is correct
A generalized version of Kauffman-Harary conjecture

Conjecture (Mathew Williamson 2007)

The minimum number of colors \( \min \text{Col}_p(D) \) of a reduced alternating virtual knot diagram \( D \) with prime determinate \( p \) is exactly the crossing number of \( D \).

- (Mathew Williamson 2007) Generalized Kauffman-Harary conjecture holds for some alternating virtual pretzel knot diagrams and alternating virtual 2-bridge knot diagrams
Kauffman-Harary conjecture and its generalization

A short review of virtual knot theory

Classical knot theory

A link diagram = a planar 4-valent graph (shadow) + “some structures on crossings”

Link types = \{all link diagrams\}/\{Reidemeister moves\}
Kauffman-Harary conjecture and its generalization

**Virtual knot theory:** Besides over crossing and under crossing, we add another structure to a crossing point: virtual crossing

![Virtual Knot Diagram](image_url)
Virtual knot types $= \{\text{all virtual knot diagrams}\}/\{\text{generalized Reidemeister moves}\}$
Kauffman-Harary conjecture and its generalization

Motivation 1

Classical knot theory:

\[ \{ S^1 \hookrightarrow S^3 \}/\text{isotopy} = \{ S^1 \hookrightarrow S^2 \times I \}/\text{isotopy} \]

Virtual knot theory:

\[ \{ S^1 \hookrightarrow \sum g \times I \}/\text{isotopy and stabilization} \]

Theorem (L. Kauffman 1999)

*Two virtual knot diagrams are equivalent if and only if their corresponding surface embeddings are stably equivalent.*
Motivation 2

Given a classic knot diagram, there is a unique Gauss diagram:

- a circle together with a chord connecting the preimages of each crossing point
- an orientation from the preimage of the overcrossing to the preimage of the undercrossing
- the writhe of the each crossing point

However NOT all Gauss diagrams are realizable as classical knot diagrams, in order to realize all Gauss diagrams, we have to add some virtual crossings.
Kauffman-Harary conjecture and its generalization

Given a virtual knot diagram, there is a unique associated Gauss diagram. However given a Gauss diagram, the corresponding virtual diagrams are not unique.

Theorem (M. Goussarov, M. Polyak, O. Viro. 2000)

A Gauss diagram uniquely defines a virtual knot isotopy class.
Theorem (Cheng)

Let $D$ be a reduced alternating virtual knot diagram with a prime determinate $p$, then each non-trivial Fox $p$-coloring of $D$ is heterogeneous.
Given a knot diagram $D$, denote the crossings and arcs of $D$ by 
\{c_1, \cdots, c_k\} and \{a_1, \cdots, a_k\} respectively. The $k \times k$ coloring matrix $M(D)$ of $D$ can be defined as below

\[
m_{ij}(D) = \begin{cases} 
2, & \text{if } a_j \text{ is the over-arc at } c_i; \\
-1, & \text{if } a_j \text{ is an under-arc at } c_i; \\
0, & \text{otherwise.}
\end{cases}
\]

If the diagram is classical then the determinate of the knot is the absolute value of the determinate of $M'(D)$, here $M'(D)$ is a $(k - 1) \times (k - 1)$ minor of $M(D)$. 
However if the diagram $D$ contains some virtual crossings, the original definition of determinate is not well-defined.

**Lemma**

*Let $D$ be a reduced alternating virtual knot diagram with $k$ classical crossings, and $M(D)$ the coloring matrix of $D$, then the absolute values of the determinates of all $(k - 1) \times (k - 1)$ minors are equal.*
Moreover, we have

**Proposition**

*Let $D$ be a reduced alternating virtual knot diagram with $k$ classical crossings, then*

1. $\det D \geq k$.

2. In additional, if $D$ is the connected sum of two reduced alternating virtual knot diagrams, say $D_1$ and $D_2$, then $\det D = \det D_1 \times \det D_2$.

The idea of proof: the determinate of $D$ equals the number of Euler circuits of $G$, here $G$ is the associated in-degree 2 out-degree 2 directed graph of $D$. 
Thank you!