Some Problems and Generalizations on Erdős-Ko-Rado Theorem

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1. Erdős-Ko-Rado Theorem

Theorem (EKR Theorem)

If $A$ is an intersecting family of $k$-subsets of $[n] = \{1, 2, \ldots, n\}$, i.e., $A \cap B \neq \emptyset$ for any $A, B \in A$, then

$$|A| \leq \binom{n-1}{k-1}$$

subject to $n \geq 2k$. Equality holds if and only if every subset in $A$ contains a common element of $[n]$ except for $n = 2k$.

1. Erdős-Ko-Rado Theorem

Theorem (EKR Theorem for Finite Vector Spaces)

If $\mathcal{A}$ is an intersecting family of $k$-dimensional subspaces of an $n$-dimensional vector space over the $q$-element field, i.e., $\dim(A \cap B) \geq 1$ for any $A, B \in \mathcal{A}$, then

$$|\mathcal{A}| \leq \left\lfloor \frac{n - 1}{k - 1} \right\rfloor$$

subject to $n \geq 2k$. Equality holds if and only if every subset in $\mathcal{A}$ contains a common nonzero vector except the case $n = 2k$.

1. Erdős-Ko-Rado Theorem

**Theorem (EKR Theorem for Permutations)**

If \( A \) is an intersecting family in \( S_n \) (the symmetric group on \([n]\)), i.e., for each pair \( \sigma, \tau \in S_n \) there is an \( i \in [n] \) with \( \sigma(i) = \tau(i) \), then

\[
|A| \leq (n - 1)!.
\]

Equality holds if and only if \( A \) is a coset of the stabilizer of a point.

- M. Deza and P. Frankl, On the maximum number of permutations with given maximal or minimal distance, JCTA 22(1977) 352-362.
A \textit{$q$-signed $k$-set} is a pair $(A, f)$, where $A \subseteq [n]$ is a $k$-set and $f$ is a function from $A$ to $[q]$. A family $\mathcal{F}$ of $q$-signed $k$-sets is \textit{intersecting} if for any $(A, f), (B, g) \in \mathcal{F}$ there exists $x \in A \cap B$ such that $f(x) = g(x)$.

Set $\mathcal{B}_n^k(q) = \{(A, f) : A \in \binom{[n]}{k}\}$ and $\mathcal{B}_n(q) = \bigcup_{i=0}^{n} \mathcal{B}_n^k(q)$.

A \textit{$r$-partial permutation} of $[n]$ is a pair $(A, f)$ with $A \in \binom{[n]}{r}$ and $f$ is an injective map from $A$ to $[n]$.

the set of all $r$-partial permutations of $[n]$ denoted by $\mathcal{P}_{r,n}$.
Theorem (EKR Theorem for Signed Sets)

(Bollobás and Leader) Fix a positive integer $k \leq n$, and let $\mathcal{F}$ be an intersecting family of $q$-signed $k$-sets on $[n]$, where $q \geq 2$. Then $|\mathcal{F}| \leq \binom{n-1}{k-1} q^{k-1}$. Unless $q = 2$ and $k = n$, equality holds if and only if $\mathcal{F}$ consists of all $q$-signed $k$-sets $(A, f)$ such that $x_0 \in A$ and $f(x_0) = \varepsilon_0$ for some fixed $x_0 \in [n]$, $\varepsilon_0 \in [q]$.

1. Erdős-Ko-Rado Theorem

Theorem (EKR Theorem for Partial Permutation)

Fix a positive integer \( r < n \), and let \( \mathcal{F} \) be an intersecting family of \( \mathcal{P}_{r,n} \). Then \( |\mathcal{F}| \leq \binom{n-1}{r-1} \frac{(n-1)!}{(n-r)!} \). Equality holds if and only if \( \mathcal{F} \) consists of all \( r \)-partial permutations \((A, f)\) such that \( i \in A \) and \( f(i) = j \) for some fixed \( i, j \in [n] \).

1. Erdős-Ko-Rado Theorem

**Theorem (Hilton, 1977)**

Let $A_1, A_2, \ldots, A_m$ be cross-intersecting families of $k$ subsets of $[n]$ with $A_1 \neq \emptyset$, i.e., for any $A_i \in A_i$ and $A_j \in A_j$, $i \neq j$, $A_i \cap A_j \neq \emptyset$. If $k \leq n/2$, then

$$\sum_{i=1}^{m} |A_i| \leq \begin{cases} \binom{n}{k}, & \text{if } m \leq n/k; \\ m\binom{n-1}{k-1}, & \text{if } m \geq n/k. \end{cases}$$

Unless $m = 2 = n/k$, the bound is attained if and only if one of the following holds:

(i) $m \leq n/k$ and $A_1 = \binom{[n]}{k}$, and $A_2 = \cdots = A_m = \emptyset$;

(ii) $m \geq n/k$ and $|A_1| = |A_2| = \cdots = |A_m| = \binom{n-1}{k-1}$.

1. Erdős-Ko-Rado Theorem

Results

The Hilton Theorem was generalized to partial permutation, signed sets and labeled sets.


Results

The Hilton Theorem was generalized to general case.

Theorem (Hilton and Milner 1967)

Let $n$ and $a$ be two positive integers with $n \geq 2a$. If $\mathcal{A}, \mathcal{B} \subseteq \binom{[n]}{a}$ with $A \cap B \neq \emptyset$ for all $A \in \mathcal{A}$ and $B \in \mathcal{B}$, then

$$|\mathcal{A}| + |\mathcal{B}| \leq \binom{n}{a} - \binom{n-a}{a} + 1.$$ 

1. Erdős-Ko-Rado Theorem

**Theorem (Frankl and Tohushige)**

Let $n$, $a$ and $b$ be three positive integers with $n \geq a + b$ and $a \leq b$. If $A \subseteq \binom{[n]}{a}$ and $B \subseteq \binom{[n]}{b}$ with $A \cap B \neq \emptyset$ for all $A \in \mathcal{A}$ and $B \in \mathcal{B}$, then

$$|A| + |B| \leq \binom{n}{b} - \binom{n-a}{b} + 1.$$ 

1. Erdős-Ko-Rado Theorem

Results

The Hilton-Milner Theorem was generalized to the general cases.

1. Erdős-Ko-Rado Theorem

**Theorem**

If \( A \subseteq \binom{[n]}{k} \) and \( B \subseteq \binom{[n]}{\ell} \) are cross-intersecting with \( k, \ell \leq n/2 \), then

\[
|A||B| \leq \binom{n-1}{k-1}\binom{n-1}{\ell-1}.
\]

Moreover, the equality holds if and only if \( A = \{A \in \binom{[n]}{k} : i \in A\} \) and \( B = \{B \in \binom{[n]}{\ell} : i \in B\} \) for some \( i \in [n] \), unless \( n = 2k = 2\ell \).

Theorem (Tokushige)

Let $p$ be a real with $0 < p < 0.114$, and let $t$ be an integer with $1 \leq t \leq 1/(2p)$. For fixed $p$ and $t$ there exist positive constants $\varepsilon$, $n_1$ such that for all integers $n, k$ with $n > n_1$ and $|\frac{k}{n} - p| < \varepsilon$, the following is true: if two families $A_1 \subset \left(\begin{array}{c} n \\ k \end{array}\right)$ and $A_2 \subset \left(\begin{array}{c} n \\ k \end{array}\right)$ are cross $t$-intersecting, then

$$|A_1||A_2| \leq \left(\frac{n-t}{k-t}\right)^2$$

with equality holding iff $A_1 = A_2 = \{ F \in \left(\begin{array}{c} n \\ k \end{array}\right) : [t] \subset F \} \ (up \ to \ isomorphism)$.

Theorem (Ellis, Friedgut, Pilpel)

For any positive integer $k$ and any $n$ sufficiently large depending on $k$, if $I, J \subset S_n$ are $k$-cross-intersecting, then $|I||J| \leq ((n - k)!)^2$. Equality holds if and only if $I = J$ and $I$ is a $k$-coset of $S_n$.

1. Erdős-Ko-Rado Theorem

**Theorem**

Let $n$ and $p$ be two positive integer with $p \geq 4$. If $\mathcal{A}$ and $\mathcal{B}$ are cross-intersecting families in $\mathcal{L}_p = \{(1, \ell_1), (2, \ell_2), \ldots, (n, \ell_n) : \ell_i \in [p], i = 1, 2, \ldots, n\}$, then

$$|\mathcal{A}||\mathcal{B}| \leq p^{2n-2},$$

and equality holds if and only if $\mathcal{A} = \mathcal{B} = \{(1, \ell_1), (2, \ell_2), \ldots, (n, \ell_n) : \ell_i = j\}$ for some $i \in [n]$ and $j \in [p]$.

2. 2-independent sets in Kneser graphs

Let $G$ be a simple graph with vertex set $V(G)$ and edge set $E(G)$. $S \subseteq V(G)$.

- $S$ is an independent set of $G$ if no two elements of $S$ are adjacent in $G$. 

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2. 2-independent sets in Kneser graphs

Let $G$ be a simple graph with vertex set $V(G)$ and edge set $E(G)$. $S \subseteq V(G)$.

- $S$ is an independent set of $G$ if no two elements of $S$ are adjacent in $G$.
- $S$ is a $k$-independent set of $G$ if $S$ can be expressed as a union of $k$ independent sets of $G$. 
Let $G$ be a simple graph with vertex set $V(G)$ and edge set $E(G)$. $S \subseteq V(G)$.

- $S$ is an independent set of $G$ if no two elements of $S$ are adjacent in $G$.
- $S$ is a $k$-independent set of $G$ if $S$ can be expressed as a union of $k$ independent sets of $G$.
- $k$-independence number $\alpha_k(G)$: the maximum size of $k$-independent sets of $G$. 

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2. 2-independent sets in Kneser graphs

The Kneser graph $K(n, k)(2k \leq n)$: vertex set $\binom{[n]}{k}$ and $A \sim B$ iff $A \cap B = \emptyset$.

- Erdős-Ko-Rado: $\alpha_1(K(n, k)) = \binom{n-1}{k-1}$. 
The Kneser graph $K(n, k)(2k \leq n)$: vertex set $\binom{[n]}{k}$ and $A \sim B$ iff $A \cap B = \emptyset$.

- Erdős-Ko-Rado: $\alpha_1(K(n, k)) = \binom{n-1}{k-1}$.
- Given a graph $G$, what is $\alpha_1(G)$?
2. 2-independent sets in Kneser graphs

Given a graph $G$, what is $\alpha_k(G)$?
Note that $A = \{A \in \binom{[n]}{k} : A \cap \{1, 2\} \neq \emptyset\}$ is a 2-independent set of $K(n, k)$.

- Someone conjectured that $\alpha_2(K(n, k)) = \binom{n-1}{k-1} + \binom{n-2}{k-1}$ for sufficient larger $n$. 
Note that $\mathcal{A} = \{A \in \binom{[n]}{k} : A \cap \{1, 2\} \neq \emptyset\}$ is a 2-independent set of $K(n, k)$.

- Someone conjectured that $\alpha_2(K(n, k)) = \binom{n-1}{k-1} + \binom{n-2}{k-1}$ for sufficient larger $n$.
- C. Godsil et al: $\alpha_2(K(9, 4)) = 95$ or 96
2. 2-independent sets in Kneser graphs

Example

\[ \mathcal{A} = \left\{ A \in \binom{[9]}{4} : 1 \in A \text{ and } A \cap \{2, 3, 4\} \neq \emptyset \right\} \]

\[ \mathcal{B} = \left\{ B \in \binom{[9]}{4} : 1 \notin A \text{ and } \{2, 3, 4\} \subset B \right\} \]

\[ \mathcal{C} = \left\{ C \in \binom{[9]}{4} : |C \cap \{5, 6, 7, 8, 9\}| \geq 3 \right\} \]

Clearly, \( \mathcal{A} \cup \mathcal{B} \) and \( \mathcal{C} \) are two disjoint independent sets of \( K(9, 4) \), and so \( \mathcal{A} \cup \mathcal{B} \cup \mathcal{C} \) is a 2-independent set.

\[ |\mathcal{A}| + |\mathcal{B}| + |\mathcal{C}| = \binom{8}{3} - \binom{5}{3} + \binom{5}{1} + \binom{5}{4} + \binom{5}{3} \binom{4}{1} = 96. \]
2. 2-independent sets in Kneser graphs

Example

\[ A = \left\{ A \in \binom{[9]}{4} : 1 \in A \text{ and } A \cap \{2, 3\} \neq \emptyset \right\} \]

\[ B = \left\{ B \in \binom{[9]}{4} : 1 \not\in A \text{ and } \{2, 3\} \subset B \right\} \]

\[ C = \left\{ C \in \binom{[9]}{4} : |C \cap \{5, 6, 7, 8, 9\}| \geq 3 \right\} \]

\( A \cup B \cup C \) is also a 2-independent set.

\[ |A| + |B| + |C| = 96. \]
2. 2-independent sets in Kneser graphs

For $n = 2k + 1$, we conjecture

$$\alpha_2(K(n, k)) = \binom{n-1}{k-1} + \binom{n-2}{k-1}$$

$$+ \begin{cases} 
\sum_{i \geq \lceil k/2 \rceil + 1} \binom{k}{i} \binom{k}{k-i}, & \text{if } k \text{ is odd;} \\
\sum_{i \geq \lceil k/2 \rceil + 1} \binom{k-1}{i} \binom{k+1}{k-i}, & \text{if } k \text{ is even.}
\end{cases}$$

$$\alpha_2(K(9, 4)) = 96.$$
2. 2-independent sets in Kneser graphs

For \( n > 2k + 1 \), we conjecture

\[
\alpha_2(K(n, k)) = \binom{n - 1}{k - 1} + \binom{n - 2}{k - 1}.
\]
3. Matching Numbers

Theorem (Turán 1941)

Let $G(V, E)$ be a graph on $n$ vertices without $k$-clique, then

$$|E| \leq \frac{(k - 2)n^2}{2(k - 1)}.$$
A hypergraph $H$ is a pair $H = (X, \mathcal{E})$ where $X$ is a set of elements, called vertices, and $\mathcal{E}$ is a set of non-empty subsets of $X$ called edges.
3. Matching Numbers

- A hypergraph $H$ is a pair $H = (X, \mathcal{E})$ where $X$ is a set of elements, called vertices, and $\mathcal{E}$ is a set of non-empty subsets of $X$ called edges.
- $H$ is said to be $k$-uniform if $|E| = k$ for all $E \in \mathcal{E}$.
A hypergraph $H$ is a pair $H = (X, \mathcal{E})$ where $X$ is a set of elements, called vertices, and $\mathcal{E}$ is a set of non-empty subsets of $X$ called edges.

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A family $\{E_1, E_2, \ldots, E_s\} \subset \mathcal{E}$ is called a matching if $E_i$’s are pairwise disjoint.
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A family $\{E_1, E_2, \ldots, E_s\} \subset \mathcal{E}$ is called a matching if $E_i$’s are pairwise disjoint.

For $\mathcal{F} \subseteq \mathcal{E}$, the matching number $\nu(\mathcal{F})$ is the size of the maximum matching contained in $\mathcal{F}$. 
Erdős posed the following conjecture.

**Conjecture**

If $\mathcal{F} \subset \binom{[n]}{k}$, $\nu(\mathcal{F}) = s$ and $n \geq (s + 1)k$, then

$$|\mathcal{F}| \leq \max \left\{ \binom{sk + k - 1}{k}, \binom{n}{k} - \binom{n - s}{k} \right\}. \quad (1)$$

The case $s = 1$ is the classical Erdős-Ko-Rado Theorem.

Erdős proved that the conjecture holds if $n > n_0(k, s)$. 
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3. Matching Numbers

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- Huang, Loh and Sudakov (2012) proved that $n_0(k, s) = 3k^2$.

The cases $k = 2$ and $k = 3$ were settled by Erdős and Gallai (1959) and Frankl (recently), respectively.
3. Matching Numbers

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- Frankl (2013) proved that $n_0(k, s) = 2(s + 1)k - s$. 

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3. Matching Numbers

- A linear path of length $\ell$ is a family of sets $\{F_1, F_2, \ldots, F_\ell\}$ such that $|F_i \cap F_{i+1}| = 1$ for each $i$ and $F_i \cap F_j = \emptyset$ whenever $|i - j| > 1$. Let $P^{(k)}_\ell$ denote the $k$-uniform linear path of length of $\ell$.

Theorem (Füredi, Jiang and Seiver)

Let $k, t$ be positive integers, where $k \geq 3$. For sufficiently larger $n$, we have

$$\text{ex}_k(n; P^{(k)}_{2t+1}) = \binom{n-1}{k-1} + \binom{n-2}{k-1} + \ldots + \binom{n-t}{k-1}.$$  

The only extremal family consists of all the $k$-sets in $[n]$ that meet some fixed set $S$ of $t$ vertices.

Theorem (Füredi, Jiang and Seiver)

Let $k, t$ be positive integers, where $k \geq 3$. For sufficiently larger $n$, we have

$$\text{ex}_k(n; \mathcal{P}_{2t+2}) = \binom{n-1}{k-1} + \binom{n-2}{k-1} + \ldots + \binom{n-t}{k-1} + \binom{n-t-2}{k-2}.$$ 

The only extremal family consists of all the $k$-sets in $[n]$ that meet some fixed set $S$ of $t$ vertices plus all the $k$-sets in $[n] \setminus S$ that contain some two fixed elements.

3. Matching Numbers

A $k$-uniform minimal cycle of length $\ell$ is a cyclic list of $k$-sets $A_1, A_2, \ldots, A_\ell$ such that consecutive sets intersect in at least one element and nonconsecutive sets are disjoint. The set of all $k$-uniform minimal cycles of length $\ell$ denoted by $C^{(k)}_\ell$. 

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A \( k \)-uniform \textit{minimal cycle} of length \( \ell \) is a cyclic list of \( k \)-sets \( A_1, A_2, \ldots, A_\ell \) such that consecutive sets intersect in at least one element and nonconsecutive sets are disjoint. The set of all \( k \)-uniform minimal cycles of length \( \ell \) denoted by \( C_\ell^{(k)} \).

A \( k \)-uniform \textit{linear cycle} of length \( \ell \), denoted by \( C_\ell^{(k)} \) is a cyclic list of \( k \)-sets \( A_1, A_2, \ldots, A_\ell \) such that consecutive sets intersect in exactly one element and nonconsecutive sets are disjoint.
3. Matching Numbers

Theorem (Füredi and Jiang)

Let $t$ be a positive integer, $k \geq 4$. For sufficiently larger $n$, we have

$$\text{ex}_k(n, C_{2t+1}^{(k)}) = \binom{n}{k} - \binom{n-t}{k}$$

and

$$\text{ex}_k(n, C_{2t+2}^{(k)}) = \binom{n}{k} - \binom{n-t}{k} + 1.$$

The only extremal family consists of all the $k$-sets in $[n]$ that meet some fixed $k$-set $S$. For $C_{2t+2}^{(k)}$, the only extremal family consists of all the $k$-sets in $[n]$ that meet some fixed $k$-set $S$ plus one additional $k$-set outside $S$.

3. Matching Numbers

Theorem (Füredi and Jiang)

Let \( t \) be a positive integer, \( k \geq 5 \). For sufficiently larger \( n \), we have

\[
ex_k(n, \binom{k}{2t+1}) = \binom{n}{k} - \binom{n-t}{k}
\]

and

\[
ex_k(n, \binom{k}{2t+2}) = \binom{n}{k} - \binom{n-t}{k} + \binom{n-t-2}{k-2}.
\]

For \( \binom{k}{2t+1} \), the only extremal family consists of all the \( k \)-sets in \([n]\) that meet some fixed \( k \)-set \( S \). For \( \binom{k}{2t+2} \), the only extremal family consists of all the \( k \)-sets in \([n]\) that meet some fixed \( k \)-set \( S \) plus all the \( k \)-sets in \([n]\) \( \setminus S \) that contain some two two fixed elements.

3. Matching Numbers

\[ S = \{ C_1 \cup \ldots \cup C_r : C_i \in C_{\ell_i}^{(k)} \text{ for } i \in [r] \} \]

**Theorem (Gu, Li and Shi)**

Let integers \( k \geq 4, r \geq 1, \ell_1, \ldots, \ell_r \geq 3 \), \( t = \sum_{i=1}^{r} \left\lfloor \frac{\ell_i + 1}{2} \right\rfloor - 1 \), and \( I = 1 \) if all the \( \ell_1, \ldots, \ell_r \) are even, \( I = 0 \) otherwise. For sufficiently large \( n \),

\[ \text{ex}_k(n; S(\ell_1, \ldots, \ell_r)) = \binom{n}{k} - \binom{n-t}{k} + I. \]

### Theorem (Gu, Li and Shi)

Let integers $k \geq 5$, $r \geq 1$, $\ell_1, \ldots, \ell_r \geq 3$, $t = \sum_{i=1}^{r} \left\lfloor \frac{\ell_i + 1}{2} \right\rfloor - 1$, and

$$J = \binom{n-t-2}{k} \text{ if all the } \ell_1, \ldots, \ell_r \text{ are even, } J = 0 \text{ otherwise.}$$

For sufficiently large $n$,

$$\text{ex}_k(n; C_{\ell_1}^{(k)}, \ldots, C_{\ell_r}^{(k)}) = \binom{n}{k} - \binom{n-t}{k} + J.$$

3. Matching Numbers

$$\mathcal{B}_n^k(q) = \{(A, f) : A \in \binom{[n]}{k}\}.$$
3. Matching Numbers

\[ \mathcal{B}_n^k(q) = \{(A, f) : A \in \binom{[n]}{k}\} . \]

**Theorem**

For positive integers \( q, n \) and \( k \) with \( k \leq n \), if \( \mathcal{F} \subset \mathcal{B}_n^k(q) \) with \( \nu(\mathcal{F}) = s \) where \( s < q \), then

\[ |\mathcal{F}| \leq sq^{k-1} \binom{n-1}{k-1} , \]

and equality holds if and only if \( \mathcal{F} \) is isomorphic to

\[ \mathcal{F}_1 = \{(A, f) \in \mathcal{B}_n^k(q) : 1 \in A, f(1) \in [s]\} . \]
3. Matching Numbers

Let $V$ be a finite set and let $p$ be an ideal of $2^V$, that is, $p$ consists of subsets of $V$ such that $B \in p$ if $B \subseteq A$ for some $A \in p$. 
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Define $\alpha_p(V) := \max\{|A| : A \in p\}$ and write

$$\mathcal{M}(V) = \{A \in p : |A| = \alpha_p(V)\}$$
3. Matching Numbers

Let $V$ be a finite set and let $\mathfrak{p}$ be an ideal of $2^V$, that is, $\mathfrak{p}$ consists of subsets of $V$ such that $B \in \mathfrak{p}$ if $B \subseteq A$ for some $A \in \mathfrak{p}$.

Define $\alpha_\mathfrak{p}(V) := \max\{|A| : A \in \mathfrak{p}\}$ and write

$$\mathcal{M}(V) = \{A \in \mathfrak{p} : |A| = \alpha_\mathfrak{p}(V)\}$$

For $H \subset V$, write $\alpha_\mathfrak{p}(H) := \max\{|A| : A \in \mathfrak{p} \text{ and } A \subseteq H\}$. 

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For $H \subset V$, write $\alpha_p(H) := \max\{|A| : A \in p \text{ and } A \subseteq H\}$.

For positive integer $s$, set

$$p = \{F \subseteq B_n^k(q) : \nu(F) \leq s\}.$$
3. Matching Numbers

To complete the proof, we only need to determine $\alpha_p(B_n^k(q))$ and $\mathcal{M}(B_n^k(q))$. 
3. Matching Numbers

Lemma

Let $V$ be a finite set and $\mathfrak{p}$ an idea on $2^V$. Suppose that there is a transitive permutation group $\Gamma$ on $V$ that keeps the ideal $\mathfrak{p}$, i.e., $\sigma(A) \in \mathfrak{p}$ for all $A \in \mathfrak{p}$ and $\sigma \in \Gamma$. Then, for each $H \subseteq V$,

$$\frac{\alpha_\mathfrak{p}(V)}{|V|} \leq \frac{\alpha_\mathfrak{p}(H)}{|H|}, \quad (2)$$

and equality holds if and only if $|H \cap S| = \alpha_\mathfrak{p}(H)$ for each $S \in \mathcal{M}(V)$. 
$\mathcal{B}_n^k(q) = \{(A, f) : A \in \binom{[n]}{k} \text{ and } f \text{ is an injection from } A \text{ to } [q]\}$

\[
\begin{array}{cccccccc}
1 & 2 & 3 & \ldots & n & 1 & 2 & 3 & \ldots & n & \ldots & 1 & 2 & 3 & \ldots & n \\
1 & 1 & 1 & \ldots & 1 & 2 & 2 & 2 & \ldots & 2 & \ldots & q & q & q & \ldots & q \\
\end{array}
\]
3. Matching Numbers

\[ \mathcal{B}_n^k(q) = \{(A, f) : A \in \binom{[n]}{k} \text{ and } f \text{ is an injection from } A \text{ to } [q]\} \]

- 1 2 3 \ldots n 1 2 3 \ldots n \ldots 1 2 3 \ldots n
- 1 1 1 \ldots 1 2 2 2 \ldots 2 \ldots q q q \ldots q

\((A, f)\) is said to be contained in the above cycle if it consists of \(k\) consecutive elements.

\[ |H| = nq \]

Determine \(\alpha_p(H)\).
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(A, f) \text{ is said to be contained in the above cycle if it consists of } k \text{ consecutive elements.}

Let \( \mathcal{H} \) be the set of (A, f) which contained in the above cycle.

Clearly \( |\mathcal{H}| = nq \).
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Let \(\mathcal{H}\) be the set of \((A, f)\) which contained in the above cycle. Clearly \(|\mathcal{H}| = nq\).

Determine \(\alpha_p(\mathcal{H})\).
Graph $G[\mathcal{H}]$: $V(G[\mathcal{H}]) = \mathcal{H}$, two vertices $(A, f)$ and $(B, g)$ are adjacent iff they are not intersecting.
3. Matching Numbers

- Graph $G[H]$: $V(G[H]) = H$, two vertices $(A, f)$ and $(B, g)$ are adjacent iff they are not intersecting.
- $G[H]$ is isomorphism to the well-known circular graph $K_{nq: k}$. 
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- $G[\mathcal{H}]$ is isomorphism to the well-known circular graph $K_{nq:k}$.
- Clique number $\omega(G)$: the maximum number of pairwise adjacent vertices in $G$. For $S \subseteq V(G)$, $\omega(G[S])$ written as $\omega(S)$. 

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- $G[\mathcal{H}]$ is isomorphism to the well-known circular graph $K_{nq:k}$.
- Clique number $\omega (G)$: the maximum number of pairwise adjacent vertices in $G$. For $S \subseteq V(G)$, $\omega (G[S])$ written as $\omega (S)$.
- $\mathcal{F}$ of $\mathcal{H}$ with $\nu (\mathcal{F}) = s$ iff $\omega (\mathcal{F}) = s$.
- To determine $\alpha_p (\mathcal{H})$ is equivalent to determine $\omega (G[\mathcal{H}])$. 
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- $K_{n:k}$: vertex set $[n]$, $i \sim j$ iff $k \leq |i - j| \leq n - k$. 

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3. Matching Numbers

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**Lemma**

Let $n$, $s$ and $k$ be three integers with $n \geq k(s + 1)$. For any vertex subset $F$ of $K_{n:k}$ with $\omega(F) = s$, then $|F| \leq ks$. Moreover, if $n > k(s + 1)$, the equality holds if and only if $F$ is the union of $s$ maximum independent sets of $K_{n:k}$. 
3. Matching Numbers

- \( \omega(G[H]) = ks \)

Let \( \mathcal{F} \) be a family of \( B_n^k(q) \) with \( \nu(\mathcal{F}) = s \). Then by the former lemmas we have

\[
\frac{|\mathcal{F}|}{|B_n^k(q)|} \leq \frac{sk}{nq}.
\]

Therefore,

\[
|\mathcal{F}| \leq \frac{sk}{nq} q^k \binom{n}{k} = sq^{k-1} \binom{n-1}{k-1}.
\]
3. Matching Numbers

For \( n \) positive numbers \( p_1, p_2, \ldots, p_n \) with \( p_1 \leq p_2 \leq \cdots \leq p_n \), let \( \mathcal{L}_p \) be the labeled \( n \)-sets given by

\[
\mathcal{L}_{n,p} = \{(i_1, i_2, \ldots, i_n) : i_j \in [p_j] \text{ for } j \in [n]\}.
\]

**Theorem**

If \( \mathcal{F} \) is a family of \( \mathcal{L}_{n,p} \) with \( \nu(\mathcal{F}) = s \leq p_1 \), then

\[
|\mathcal{F}| \leq sp_2p_3\cdots p_n,
\]

and equality holds if and only if \( \mathcal{F} = \{(i_1, i_2, \ldots, i_n) \in \mathcal{L}_{n,p} : i_j \in S\} \) for one \( s \)-subset \( S \) of \([p_1]\) and one \( j \) of \([n]\) with \( p_j = p_1 \).
Theorem

Let $n$ and $s$ be two positive integers with $s \leq n$. If $\mathcal{F}$ is a family of $S_n$ with $\nu(\mathcal{F}) = s$, then $|\mathcal{F}| \leq s(n - 1)!$. 
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Theorem

Let $n$ and $s$ be two positive integers with $s < n$. If $\mathcal{F}$ is a family of $\mathcal{P}_{r,n}$ with $\nu(\mathcal{F}) = s$, then $|\mathcal{F}| \leq s \binom{n-1}{r-1} \frac{(n-1)!}{(n-r)!}$.
Many Thanks!