Convexity in examples, from Lie theory, symplectic geometry and integrable Hamiltonian systems

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Based on joint work with Tudor Ratiu done at SJTU (99 pages submitted):

- T. Ratiu, Ch. Wacheux, NTZ, *Convexity of singular affine structures and toric-focus integrable Hamiltonian systems*, arXiv:1706.01093 (77 pages)

I’ll give a series of interesting examples which are relatively easy to imagine, so that students and non-experts can understand, then indicate the theories behind them and generalizations.

Many thanks to the School of Mathematics and the colleagues, secretaries and students here for the invitation, warm hospitality and excellent working conditions, especially Prof. Tudor Ratiu, Prof. Jiangsu Li and Ms. Jie Hu, and also Prof. Yaokun Wu, Prof. Tongsuo Wu, Prof. Xiang Zhang, Ms. Jie Zhou, Ms. Shi Yi, ...
Outline of the talk

1. Gorilla selling bananas: a convex math puzzle
2. Schur-Horn theorem and generalizations
3. Local-global convexity principle
4. Non-linear convexity theorems
5. Convexity in groupoid setting
6. What do we need for convexity?
7. Toric varieties and momentum polytopes
8. Toric-focus integrable Hamiltonian systems
9. Integral affine black holes
10. Positive results on convexity with monodromy
Example 1: Gorilla selling bananas

A nice puzzle for every one:
A gorilla has 3000 bananas. He wants to bring them to the market, which is 1000 km away, to sell. Each time he can carry at most 1000 bananas, and he has to eat 1 banana per every km he goes. What is the maximal number of bananas that he can bring to the market?

Note: He can drop bananas midway, no one will steal them, and they won’t spoil. No one will help him either.

Where is convexity? Try to figure out!
- Linear inequalities (constraints)
- Convex optimization (a linear function to optimize on a polytope)
- Convexity = a bunch of linear inequalities.
Inspirational math books for children

• Detailed solution to ”Gorilla selling bananas” is given in the book ”Math lessons for Mirella”, which I wrote and published by Sputnik Education, of which I’m a founder.

• **Sputnik Education** started publishing inspirational math books for children since 2015, and has published more than 30 books (original, or translated from other languages including English, Russian, Portuguese), more than 100 thousand copies.

• **Newsletter of the European Mathematical Society** has a 3-page article in Dec. 2016 issue about us.

• We’re looking for international cooperation, in particular with China!
Some original books from Sputnik Education

Maths and Arts, Romeo searching for the Princess, Math Olympics, Problems in Algebra and Arithmetics, Combinatorial Geometry, etc.
Example 2: Schur-Horn theorem

**Theorem (Schur (1923): inclusion – Horn (1954): equality)**

The set of diagonals of an isospectral set of Hermitian $n \times n$ matrices, viewed as a subset of $\mathbb{R}^n$, is equal to the convex polytope whose vertices are the vectors formed by the $n!$ permutations of its eigenvalues.

Example: Eigenvalues are 1, 3, 7 and diagonal is $(d_1, d_2, d_3)$, then $(d_1, d_2)$ lies in the convex hexagon in the picture.

**Consequences of convexity?**
Optimization, Combinatorics (counting points, volume etc.), Topology (Morse theory, computation of cohomology), etc.

**Generalizations?** Lie theory, symplectic geometry, infinite-dimensional generalizations etc.
Generalizations of Schur–Horn

**Theorem (Kostant 1973, Linear convexity theorem in Lie theory)**

The projection of a coadjoint orbit of a connected compact Lie group relative to a bi-invariant inner product onto the dual of a Cartan subalgebra is the convex hull of an orbit of the Weyl group.

Schur–Horn is a particular case of linear Kostant


Let \((M^{2n}, \omega)\) be a 2n-dimensional symplectic manifold endowed with a Hamiltonian \(\mathbb{T}^k\)-action with momentum map \(J : M \to \mathbb{R}^k\). Then the fibers of \(J\) are connected and \(J(M)\) is a compact convex polytope, namely the convex hull of the image of the fixed point set of the \(\mathbb{T}^k\)-action.

Linear Kostant is a particular case of Atiyah–Guillemin–Sternberg: symplectic manifold = coadjoint orbit with Kirillov-Kostant-Souriau form, momentum map = projection to the dual of Cartan torus.
• **Symplectic manifolds**: appear in physics (cotangent bundles, phase space of Hamiltonian systems), Lie theory (e.g., coadjoint orbits of Lie algebras), geometry (e.g., Kähler manifolds) etc. \((M, \omega)\) called symplectic if \(\omega\) is a nondegenerate closed 2-form on \(M\). Then for each function \(f\) on \(M\) there is a unique **Hamiltonian vector field** \(X_f\) defined by

\[
X_f \cdot \omega = df
\]

• **Momentum map** \(J = (J_1, \ldots, J_k) : (M, \omega) \to \mathbb{R}^k\) of a Hamiltonian torus action means that \((X_{J_1}, \ldots, X_{J_k})\) are generators of a \(T^k\) on \(M\).

• The case of Hamiltonian actions of non-Abelian compact groups:

**Theorem (Kirwan 1984)**

*Hamiltonian action of a compact group on a compact symplectic manifold with an equivariant momentum map. Then the intersection of the image of the momentum map with a Weyl chamber is a convex polytope.*
Other developments and generalizations

- Involution, ”real” convexity: Kostant’s theorem for real flag manifolds (1973), Duistermaat (1983),
- and so on (see Overview in Ratiu-Wacheux-Zung 2017)
Example 3: Tietze–Nakajima theorem

The **local-global convexity principle** is one of the main tools in the study of convexity. Its origins go back to the following theorem:

**Theorem (Tietze 1928, Nakajima 1928)**

Let $C$ be a **closed** set in $\mathbb{R}^n$. Then $C$ is **convex** if and only if it is **connected** and **locally convex**.

(Local convexity means that every point admits a convex neighborhood).

This theorem is easy to prove. I gave it as an *exercise in elementary topology* for my 3rd year undergrad students.

Note: Without connectedness, a set cannot be convex. Without closedness, the theorem is also false. For example, the figure without point $C$ is non-convex but locally convex.
- Tietze–Nakajima local-global convexity principle admits many versions and generalizations over the last century, also in infinite dimensions.
- Condeveaux–Dazord–Molino (1988) were the first to use it (instead of Morse theory) to give an elegant simple proof of symplectic convexity theorems.
- Hilgert–Neeb–Planck (1994) gave a version of it well adapted for symplectic convexity. Since then, it became a very important tool in convexity. In particular, Flashka–Ratiu (1996) needed it to prove convexity for compact Poisson Lie groups (Morse theory didn’t work there).
- The following simple version also works very well for symplectic convexity:

**Lemma (Local-global convexity lemma, Z 2006)**

Let $X$ be a connected locally convex regular affine manifold with boundary, and $\phi : X \to \mathbb{R}^m$ a proper locally injective affine map. Then $\phi$ is injective and its image $\phi(X)$ is convex in $\mathbb{R}^m$. 
Example 4: Conjugacy classes in a compact Lie group

\( K = \text{simple compact group}, \ T = \text{maximal torus in} \ K, \ N_T = \text{normalizer of} \ T \text{ in} \ K, \ \mathcal{W} = N_T / T = \text{Weyl group}, \ \Lambda = \ker \exp : \mathfrak{t} \to T = \text{cocharacter lattice}. \) Then the set of conjugacy classes \( K/K = T/\mathcal{W} \cong \mathfrak{t}/\tilde{\mathcal{W}} \) where \( \tilde{\mathcal{W}} \ltimes \mathcal{W} \times \Lambda \) is the affine Weyl group. A fundamental domain of the action of \( \tilde{\mathcal{W}} \) on \( \mathfrak{t} \) is a simplex in \( \mathfrak{t} \) called a Weyl alcove \( \Delta \). The natural bijection from \( \Delta \) to the set of conjugacy classes \( K/K \) is given by the exponential map (which is non-linear).

Weyl alcove \( \Delta \) for the case SU(3). Picture borrowed from M. Thaddeus.
Example 5: Multiplicative Horn problem

Problem: What is the shape of the set

\[ P = \{ ([A], [B], [C]) \in \Delta^3 \mid A, B, C \in K; ABC = 1 \} \]

(Additive Horn problem: Klyashko, Knutson–Tao; multiplicative problem: Belkale–Ressayre–Agnihotri–Woodward ... in 1990s)

The case \( K = SU(2) \): \( P \) is a regular tetrahedron inside a cube. Picture borrowed from M. Thaddeus.
Other non-linear convexity results


Weyl (1949), Horn (1954). Let $P$ be the set of positive definite Hermitian matrices whose determinant equals 1 and $\Sigma_\lambda$ the isospectral subset of matrices in $P$ defined by the given eigenvalues $(\lambda_1, \ldots, \lambda_n) \in \mathbb{R}^n$. The image of the map

$$P \ni p \mapsto (\log(\det p_1), \ldots, \log(\det p_n)) \in \mathbb{R}^n,$$

where $p_k = (p_{ij})_{i,j=1,\ldots,k}$, is a convex polytope.

- Kostant non-linear convexity theorem (1973) = Lie-theoretical generalization of Weyl-Horn theorem. Let $G$ be a connected semisimple Lie group and $G = KAN = PK$ its Iwasawa and Cartan decomposition, respectively, with $A \subset P$. Let $O_a$ be the $K$-orbit of $a \in A$ in $P$ (by conjugation) and $\rho_A : G \to A$ the Iwasawa projection $\rho_A(kan) = a$. Identify $A$ with its Lie algebra $\mathfrak{a}$ by the exponential map. Then $\rho_A(O_a)$ is the convex hull of the Weyl group orbit through $a$.  

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Other non-linear convexity results


- Zung (2006): Linearization of proper quasi-symplectic groupoids \((\Gamma \rightrightarrows P, \omega + \Omega)\) and convexity of their Hamiltonian actions. This result contains many other linear and nonlinear convexity results as special cases.
Theorem (Z 2006)

Let \((M, \sigma)\) be a connected quasi-Hamiltonian manifold of a proper quasi-symplectic groupoid \((\Gamma \rightrightarrows P, \omega + \Omega)\) with coad-connected isotropy groups, with a proper momentum map \(\mu\). Assume that the orbit space \(P/\Gamma\) of \(\Gamma\) is simply-connected, and denote by \(j : P/\Gamma \to \mathbb{R}^k\) an integral affine immersion from \(P/\Gamma\) to \(\mathbb{R}^k\). Assume that at least one of the following additional conditions is satisfied:

1) \(M\) is compact.
2) \(j\) is an embedding and \(j(P/\Gamma)\) is closed in \(\mathbb{R}^k\).
3) \(j\) is an embedding and \(j(P/\Gamma)\) is convex in \(\mathbb{R}^k\).

Then the transverse momentum map \(\hat{\mu}\) and the composed map \(j \circ \hat{\mu}\) are injective, and the image \(j \circ \hat{\mu}(\hat{M}) = j(\mu(M)/\Gamma)\) is a convex subset in \(\mathbb{R}^k\) with locally polyhedral boundary. (We don’t count boundary points which lie in the closure of \(j(\mu(M)/\Gamma)\) but not in \(j(\mu(M)/\Gamma)\)). In particular, \(\hat{M}\) with its integral affine structure is isomorphic to a convex subset of \(\mathbb{R}^k\) with locally polyhedral boundary.
Theorem (Weinstein 2001)

For any positive-definite quadratic Hamiltonian function $H$ on the standard symplectic space $\mathbb{R}^{2k}$, denote by $\phi(H)$ the $k$-tuple $\lambda_1 \leq \ldots \leq \lambda_k$ of frequencies of $H$ ordered non-decreasingly, i.e. $H$ can be written as $H = \sum \lambda_i (x_i^2 + y_i^2)/2$ in a canonical coordinate system. Then for any two given positive nondecreasing $n$-tuples $\lambda = (\lambda_1, \ldots, \lambda_k)$ and $\gamma = (\gamma_1, \ldots, \gamma_k)$, the set

$$\Phi_{\lambda, \gamma} = \{ \phi(H_1 + H_2) \mid \phi(H_1) = \lambda, \phi(H_2) = \gamma \}$$

(1)

is a closed, convex, locally polyhedral subset of $\mathbb{R}^k$.

Remark: The above set $\Phi_{\lambda, \gamma}$ is closed but not bounded. For example, when $k = 1$ then $\Phi_{\lambda, \gamma}$ is a half-line. Weinstein’s theorem can be recovered from proper symplectic groupoid setting.
What do we need for "symplectic convexity"?

- Intrinsic/transversal affine structure, which comes from the quasi/twisted/pre symplectic structure.
- Local convexity, often obtained by local normal form theorems, e.g., Guillemin–Sternberg–Marle normal form for a symplectic tube of a group action), or Weinstein–Zung linearization for proper groupoids.
- Some method to go from local to global, e.g. Morse theory, but especially the local-global convexity principle.
- Other auxiliary technical results, e.g., connectedness, openness or closedness of the momentum maps, symplectic involution (for dealing with "real" convexity)
- A clear definition of what does it mean to be convex in singular or non-globally-flat cases (where things cannot be embedded into a vector space).
Where does the affine structure come from?

Period integral over generating 1-form $\alpha$ or (twisted/pre) symplectic 2-form $\omega$

In the picture: $Q_c, Q_{c'}$ are level sets (of momentum map or singular fibration), $\gamma_c, \gamma_{c'}$ are 1-cycles with homotopy cylinder $\Sigma_{c,c'}$. Affine function:

$$F(c) = \int_{\Sigma_{c,c'}} \omega = \int_{\gamma_c} \alpha - \int_{\gamma_{c'}} \alpha$$

This is called Action function formula: Einstein (1917, Bohr-Sommerfeld quantization), Mineur (1935, proof of action-angle variables), Arnold, ...
Example 7: Delzant polytopes and compact toric manifolds

Compact toric manifold = compact manifold $M$ admitting a Hamiltonian action of $\mathbb{T}^n$ where $n = \dim M/2$. Image of the momentum map is a convex polytope (by Atiyah–Guillemin–Sternberg) which satisfies 3 properties: rationality (facets are given by linear equations with integral linear coefficients), simplicity, and regularity. Compact (symplectic or Kähler) toric manifolds are classified by such polytopes up to isomorphisms (Delzant 1988).

Example: Delzant polytopes for Hirzebruch surfaces: $\mathbb{CP}^1 \times \mathbb{CP}^1$, $\mathbb{CP}^2 \# \overline{\mathbb{CP}^2}$, etc.
- Problem: What about convex polytopes which are **not regular**, or **not simple**, or **not rational**?
Example 8: Gelfand-Cetlin polytope

Fix a spectrum $\lambda = (\lambda_1 \geq \cdots \geq \lambda_n)$. For any Hermitian matrix with given spectrum $A \in O(\lambda)$, and any integer $1 \leq k \leq n$, denote by $\gamma_{1,k}(A) \geq \cdots \geq \gamma_{k,k}(A)$ the eigenvalues of the upper-left $k \times k$ submatrix of $A$. Then $\Lambda = \{\gamma_{i,j}\}$ satisfies the Gelfand–Cetlin inequalities

$$
\begin{array}{cccccccc}
\lambda_1 & \geq & \lambda_2 & \geq & \lambda_3 & \geq & \cdots & \geq & \lambda_{n-1} & \geq & \lambda_n \\
\gamma_{1,n-1} & \succ & \gamma_{2,n-1} & \succ & \gamma_{3,n-1} & \succ & \cdots & \succ & \gamma_{n-1,n-1} \\
\gamma_{1,n-2} & \succ & \gamma_{2,n-2} & \succ & \cdots & \succ & \gamma_{n-2,n-2} \\
\gamma_{1,2} & \succ & \gamma_{2,2} \\
\gamma_{1,1}
\end{array}
$$

Moreover, these functions commute pairwise and form on $O(\lambda)$ and form an integrable Hamiltonian system (Guillemin–Sternberg 1983). $O(\lambda)$ is not toric but admits a toric degeneration. The polytope is not simple. The singular fibers of the system are smooth! (Bouloc–Miranda–Zung 2017)
Integrable Hamiltonian systems in the sense of Liouville

- \( H : (M, \omega) \to \mathbb{R} \): a Hamiltonian function on a symplectic manifold of dimension \( 2n \) \((n \geq 1) \) is called the **degree of freedom**.

- \( H_1 = H \) is automatically a first integral of the system. Integrability a la Liouville means that there exist \( n - 1 \) additional commuting first integrals \( H_2, \ldots, H_n \) such that the **momentum map**
  \[
  H = (H_1, \ldots, H_n) : M^{2n} \to \mathbb{R}^n
  \]
is of rank \( n \) (i.e. the functions \( H_1, \ldots, H_n \) are independent) a.e.

- **Base space** \( B = \{ \text{connected level sets of the momentum map} \ H = (H_1, \ldots, H_n) : M \to \mathbb{R}^n \} \). \( B \) admits a natural singular **integral affine structure**, due to the existence of **action-angle variables**
  \[
  (D^n \times \mathbb{T}^n, \omega = \sum dp_i \wedge dq_i)
  \]

- Problem: what about the **convexity of** \( B \)?
Singularities of integrable Hamiltonian systems

Most singular points of the momentum map $H : M^{2n} \to \mathbb{R}^n$ are nondegenerate; they can be linearized locally (Williamson, Rüssmann, Vey, Eliasson) or near a compact orbit (Miranda–Zung).

Theorem (Local linearization, Vey–Eliasson)

If $p \in M^{2n}$ is a non-degenerate singular point of an integrable Hamiltonian system $F = (F_1, \ldots, F_n) : M \to \mathbb{R}^n$, then there exist local symplectic coordinates $(x_1, \ldots, x_n, \xi_1, \ldots, \xi_n)$ about $p$, such that $\{F_i, q_j\} = 0$, for all $i, j$, where

- $q_i = e_i = x_i^2 + \xi_i^2$ $(1 \leq i \leq k_e)$ are elliptic components,
- $q_{k_e+i} = h_i = x_{i+k_e} \xi_{i+k} (1 \leq i \leq k_h)$ are hyperbolic components,
- \[
\begin{cases}
q_{2i-1+k_e+k_h} = f^1_i = x_{2i-1+k_e+k_h} \xi_{2i+k_e+k_h} - x_{2i+k_e+k_h} \xi_{2i-1+k_e+k_h} \\
q_{2i+k_e+k_h} = f^2_i = x_{2i-1+k_e+k_h} \xi_{2i-1+k_e+k_h} + x_{2i+k_e+k_h} \xi_{2i+k_e+k_h}
\end{cases}
\]
  $(1 \leq i \leq k_f)$ are focus-focus components,
- $q_{k+i} = x_i (1 \leq i \leq n - \kappa)$ are regular components.
Theorem (Zung 1996)

\( N \) = non-degenerate singular fiber of corank \( \kappa \) and Williamson type \( \mathbb{k} = (k_e, k_h, k_f) \) in an integrable Hamiltonian system given by a proper momentum map \( H : M^{2n} \rightarrow \mathbb{R}^n \). Then \( \exists \) neighborhood \( \mathcal{U}(N) \) of \( N \) in \( M^{2n} \), saturated by the fibers of the system, such that:

(i) \( \exists \) an effective Hamiltonian action of \( \mathbb{T}^{k_e+k_f+(n-\kappa)} \) on \( \mathcal{U}(N) \) which preserves the system. This number \( k_e + k_f + (n - \kappa) \) is maximal possible.

(ii) \( (\mathcal{U}(N), \text{associated Lagrangian torus fibration}) \) is homeomorphic to the quotient of a direct product of elementary non-degenerate singularities and a regular Lagrangian torus foliation of the type

\[
(\mathcal{U}(\mathbb{T}^{n-\kappa}), \mathcal{L}^r) \times (P^2(N_1^e), \mathcal{L}_1^e) \times \cdots \times (P^2(N_{k_e}^e), \mathcal{L}_{k_e}^e) \times \\
\times (P_h^2(N_1^h), \mathcal{L}_1^h) \times \cdots \times (P_h^2(N_{k_h}^h), \mathcal{L}_{k_h}^h) \times (P^4(N_1^f), \mathcal{L}_1^f) \times \cdots \times (P^4(N_{k_f}^f), \mathcal{L}_{k_f}^f)
\]

by a free action of a finite group \( \Gamma \).
A **toric-focus system** is an integrable Hamiltonian system whose singularities are nondegenerate and have no hyperbolic component, only elliptic and/or focus-focus components. Why toric-focus?

- Can be found everywhere in physical systems, e.g.: spherical pendulum, Lagrange top, spin systems, focusing NLS equation, Jaynes–Cummings–Gaudin, etc. Related also to Lagrangian fibrations on Calabi–Yau (mirror symmetry), tropical affine structures.
- Base spaces are still manifolds (with hyperbolic singularities the base spaces are not manifolds). The integral affine structure has **focus singularities**, but one can still talk about convexity.
- Special cases: semi-toric (additional condition on torus actions).
Monodromy formula around focus singularities

\[
\begin{pmatrix}
\gamma_1^{new} \\
\gamma_2^{new}
\end{pmatrix} =
\begin{pmatrix}
1 & k \\
0 & 1
\end{pmatrix}
\begin{pmatrix}
\gamma_1 \\
\gamma_2
\end{pmatrix}.
\]

\(k = \text{index}\) of the focus singularity (the case with only 1 focus component)
Focus singularity on the base space

The 2D case: Multi-valued affine coordinate system \((F, G)\): \(F\) is single-valued, \(G\) has 2 branches \(G_l\) and \(G_r\):

\[
G_r = G_l + kF \quad \text{when} \quad F > 0; \quad G_r = G_l \quad \text{when} \quad F \leq 0.
\]

Related to Duistermaat–Heckman formula w.r.t. Hamiltonian \(\mathbb{T}^1\) near a focus-focus singular fiber. Convex because \(k > 0\) (would be non-convex if \(k < 0\))
Focus singularity on the base space

The higher-dimensional case (with 1 focus-focus component):

Focus submanifold $S$ of codimension 2 is curved in general but lies on a flat $(n - 1)$-dimensional subspace.
Straight lines and convexity

- We will say that our integral affine structure is **convex** if between any two points there is a **straight line** joining them.

- Problems due to focus singularities:
  - Straight lines may be non-unique, or may be non-existent (even when the base space is homeomorphic to a ball)
  - A straight line may be singular (it goes through focus singularities): extension problem when hitting a focus point

- **Singular straight line** = limit of a family of regular straight lines.

- **Branching** at focus points. Up to $2^k$ branches if $k$ focus components.

![Diagram](image-url)
Local convexity near a focus singularity

Two potential straight lines $\gamma_l$ and $\gamma_r$ from $A$ to $B$ (which might be broken).

The equations for the points of $\gamma_l : [0, 1] \rightarrow \text{Box}$ are

$$ F(\gamma_l(t)) = tF(B) + (1 - t)F(A) \quad ; \quad G_l(\gamma_l(t)) = tG_l(B) + (1 - t)G_l(A) $$

and the equations for the points of $\gamma_r : [0, 1] \rightarrow \text{Box}$ are

$$ F(\gamma_r(t)) = tF(B) + (1 - t)F(A) \quad ; \quad G_r(\gamma_r(t)) = tG_r(B) + (1 - t)G_r(A) $$

At least one of the two $\gamma_l$ and $\gamma_r$ is not broken. The same situation in higher dimensions (with only 1 focus component).
Challenge posed by monodromy

Phenomenon: Possible **loss of convexity** due to complicated monodromy (created by many focus points, or higher order focus points, or two disjoint focus curves, etc.)

- Local-global convexity principle no longer valid when the monodromy is complicated.
- Existence of non-convex integral affine $S^2$ (which is locally convex).
- Non-convexity examples near focus$^2$ points.
- Non-convexity examples in 3D with two focus curves.
- Under some natural additional conditions, there are still positive global convexity results.
Example 9: integral affine black-hole and non-convex $S^2$

Glueing a shuriken into a flower. This flower is an ”affine blackhole”: the rays from the center $A$ cannot get out of the flower.
Example 9: integral affine black-hole and non-convex $S^2$

Glue the blackhole flower with an appropriate convex octagon to get a non-convex $S^2$;

Remark: There are also examples of convex integral affine $S^2$. So monodromy may lead to non-convexity but is not a total obstruction.
A non-convex 4D situation with a focus point

In the following picture, all the 4 potential straight lines from A to B turn out to be broken lines, so there is no straight line from A to B. Local coordinate system \((F_1, H_1, F_2, H_2)\), where \(F_1, F_2\) are integral affine.
A non-convex 3D situation with 2 focus curves

In the following picture, all the 4 potential straight lines from A to B also turn out to be broken lines, so there is no straight line from A to B.
Theorem (Ratiu–Wacheux–Z 2017)

$\mathcal{B} = 2D$ locally-convex base space with non-empty boundary of a toric-focus system on a connected compact symplectic $M^4$ (with or without boundary). Then $\mathcal{B}$ is convex. Moreover, if $\mathcal{B}$ is orientable, then it is a disk or an annulus. If $\mathcal{B}$ is an annulus, there is a global single-valued non-constant affine function $F$ on $\mathcal{B}$ such that the boundary components of $\mathcal{B}$ are straight curves on which $F$ is constant.

Global convexity of 2D focus-toric systems: proper case

Theorem

Let $B$ be the 2-dimensional base space of a toric-focus integrable Hamiltonian system on a connected, non-compact, symplectic, 4-manifold without boundary. Assume:

(i) the system has elliptic singularities (i.e., the boundary of $B$ is not empty);
(ii) the number of focus points in $B$ is finite and the interior of $B$ is homeomorphic to an open disk;
(iii) $B$ is proper.

Then $B$ is convex (in its own underlying affine structure).

Remark: Without the properness condition the theorem would fail (Pelayo–Ratiu–Vu Ngoc: cartography of different proper and non-proper semi-toric systems).
Global convexity of semi-toric systems in higher dimensions

**Theorem**

Let $\mathcal{B}$ be the base space of a toric-focus integrable Hamiltonian system with $n$ degrees of freedom on a connected compact symplectic manifold $M$. Assume that the system admits a global Hamiltonian $\mathbb{T}^{n-1}$-action. Then $\mathcal{B}$ is convex.

**Theorem**

Let $\mathcal{B}$ be the $n$-dimensional base space of a toric-focus system on a connected, non-compact, symplectic, $2n$-manifold without boundary s.t.:

(i) The system admits a global Hamiltonian $\mathbb{T}^{n-1}$-action;
(ii) the set of focus points in $\mathcal{B}$ is compact;
(iii) the interior of $\mathcal{B}$ is homeomorphic to an open ball in $\mathbb{R}^n$;
(iv) $\mathcal{B}$ is proper.

Then $\mathcal{B}$ is convex (in its own underlying affine structure).
THANK YOU!