SYMBOLIC DYNAMICS AND THE STABLE ALGEBRA OF MATRICES

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Abstract. Part two of the second half of lectures on symbolic dynamics and the stable algebra of matrices.

Lecture 5 - A (very) brief introduction to some algebraic K-theory
Lecture 6 - Strong shift equivalence theory over a ring
Lecture 7 - Automorphisms of shifts of finite type
Lecture 8 - Wagoner’s SSE complexes and some applications

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1. Lecture 7 - Automorphisms of shifts of finite type

We turn now to the study of automorphisms of shifts of finite type. In general, an automorphism of any dynamical system is simply a self-conjugacy of the given system. The investigation of automorphisms (and the group of automorphisms) of shifts of finite type (and more generally, of subshifts) is itself a very active and important topic in symbolic dynamics.

It is perhaps unsurprising that, even in the context of the classification problem, the study of automorphisms plays an important role. Part of this role is indirect: various tools and ideas which were originally used to study automorphism groups (see e.g. sign-gyration, in lecture 4) in fact turned out to play key roles in understanding the conjugacy problem. In addition to this however, certain parts of the study of the automorphism groups (notably, the dimension representation) even appear directly, for example in methods that construct counterexamples to Williams’ conjecture (as well as other areas). Some of this we will discuss in lecture 4.

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Finally, let us set some notation, and make a standing assumption. For a matrix $A$ over $\mathbb{Z}_+$, we will let $(X_A, \sigma_A)$ refer to the shift of finite type defined as the edge shift (as defined by Mike) corresponding to the graph associated to $A$ (i.e. the graph $\Gamma_A$, also as defined by Mike). To focus on the important case, and to avoid repeatedly writing the same phrase, we will only consider shifts of finite type $(X_A, \sigma_A)$ where $A$ is primitive.

Beginning at the beginning, what do we mean by an automorphism?

**Definition 1.1.** Let $(X, \sigma)$ be a shift of finite type. An automorphism of $(X, \sigma)$ is a homeomorphism $\alpha: X \to X$ such that $\alpha \sigma = \sigma \alpha$. The collection of automorphisms of $(X, \sigma)$ forms a group (under composition) which we call the group of automorphisms of $(X, \sigma)$, and we denote this group by $\text{Aut}(\sigma)$.

In other words, an automorphism of $(X, \sigma)$ is simply a self-conjugacy of the system $(X, \sigma)$, and the automorphism group is the group of all self-conjugacies. Of course, the definition above makes sense in the more general category of dynamical systems; this lecture concerns only the case of shifts of finite type (which is already quite interesting!).

**Examples:**

1. The shift $\sigma$ is itself an automorphism of $(X, \sigma)$, i.e. $\sigma \in \text{Aut}(\sigma)$.
2. Let $(X_3, \sigma_3)$ denote the full shift on the symbol set $\{0, 1, 2\}$ and define an automorphism $\alpha \in \text{Aut}(\sigma_3)$ given by the block code
   
   $$\alpha_0: x \mapsto x + 1 \text{ mod } 3, \quad x \in \{0, 1, 2\}.$$

   Thus for example, $\alpha$ acts like the following:

   $\begin{array}{cccc}
   0 & 1 & 0 & 2 \\
   \downarrow \alpha & \downarrow \alpha & \downarrow \alpha & \downarrow \alpha \\
   1 & 2 & 1 & 0 \\
   \end{array}
   \begin{array}{cccc}
   0 & 1 & 0 & 2 \\
   0 & 0 & 0 & 1 \\
   1 & 0 & 1 & 2 \\
   \end{array}
   \begin{array}{cccc}
   0 & 1 & 0 & 2 \\
   0 & 0 & 0 & 1 \\
   1 & 0 & 1 & 2 \\
   \end{array}$

   By the Curtis-Hedlund-Lyndon Theorem (see Mike lectures), any automorphism is induced by a block code. This leads immediately to the following observation:

   **Observation:** If $(X, \sigma)$ is a shift of finite type, then $\text{Aut}(\sigma)$ is a countable group.

   It turns out that, although it is countable, the group $\text{Aut}(\sigma)$ is quite complicated. The example (2) above was induced by a 0-block code, but there are automorphisms defined by block codes that need a large range (given a shift $(X, \sigma)$ and a range $R$, there are only finitely many automorphisms in $\text{Aut}(\sigma)$ having range $\leq R$). To give an
indication that $\text{Aut}(\sigma)$ is quite large when $(X, \sigma)$ is a shift of finite type, consider the following two theorems regarding the structure of subgroups of $\text{Aut}(\sigma)$.

**Theorem 1.2.** Let $(X_A, \sigma_A)$ be a shift of finite type where $A$ is a primitive matrix.

1. (Boyle-Lind-Rudolph in [2]) The group $\text{Aut}(\sigma_A)$ contains isomorphic copies of each of the following groups:
   a. Any finite group.
   b. $\bigoplus_{i=1}^{\infty} \mathbb{Z}$.
   c. The free group on two generators $\mathbb{F}_2$.

2. (Kim-Roush in [3]) For any $n \geq 2$, let $(X_n, \sigma_n)$ denote the full shift on $n$ symbols. Then $\text{Aut}(\sigma_n)$ is isomorphic to a subgroup of $\text{Aut}(\sigma_A)$.

In particular, by part (1), $\text{Aut}(\sigma_A)$ is never amenable. By part (2), for full shifts, the isomorphism types of groups that can appear as subgroups of $\text{Aut}(\sigma_n)$ is independent of $n$.

We do not have a good understanding of what types of countable groups can be isomorphic to a subgroup of the automorphism group of a full shift on $n$ symbols (for any $n$) (we’ll give some conditions later, but we have very few necessary conditions).

### 1.1. Simple Automorphisms.

Part (1), and most other classical constructions of automorphisms, are built using so-called marker methods. In [11], Nasu introduced a class of automorphisms, known as simple automorphisms, which contains the collection of marker constructions in it, and defines an important subgroup of $\text{Aut}(\sigma)$.

Let $A$ be a square matrix over $\mathbb{Z}_+$, and let $\Gamma_A$ be its associated directed graph. A **simple graph automorphism** of $\Gamma_A$ is a graph automorphism of $\Gamma_A$ which fixes all vertices. Any simple graph automorphism induces a corresponding automorphism $\gamma \in \text{Aut}(\sigma_A)$ given by a 0-block code, and we call an automorphism $\alpha \in \text{Aut}(\sigma_A)$ simple if

$$\alpha = \Psi^{-1} \gamma \Psi$$

where $\Psi: (X_A, \sigma_A) \to (X_B, \sigma_B)$ is a conjugacy to some shift of finite type $(X_B, \sigma_B)$ and $\gamma \in \text{Aut}(\sigma_B)$ is induced by a simple graph automorphism of $\Gamma_B$.

For example, the graph automorphism defined by permuting the edges $c$ and $d$ of the graph
is a simple graph automorphism.

Now we can define

\[ \text{Simp}(\sigma_A) = \text{the subgroup of } \text{Aut}(\sigma_A) \text{ generated by simple automorphisms.} \]

It is immediate to check that Simp(\(\sigma_A\)) is a normal subgroup of Aut(\(\sigma_A\)).

**Example:** There is a conjugacy from the full 3-shift \((X_3, \sigma_3)\) on symbols \(\{0, 1, 2\}\) to the edge shift of finite type \((X_A, \sigma_A)\) presented by the graph given above, on symbol set \(\{a, b, c, d, e, f\}\). Here the matrix \(A\) is given by \(A = \begin{pmatrix} 2 & 2 \\ 1 & 1 \end{pmatrix}\), and a conjugacy

\[ \Psi: (X_3, \sigma_3) \to (X_A, \sigma_A) \]

is given by the block code:

\[
\begin{align*}
00 & \mapsto a & 10 & \mapsto b & 20 & \mapsto f \\
01 & \mapsto a & 11 & \mapsto b & 21 & \mapsto f \\
02 & \mapsto d & 12 & \mapsto c & 22 & \mapsto e 
\end{align*}
\]

with inverse given by

\[
\begin{align*}
a & \mapsto 0 & d & \mapsto 0 \\
b & \mapsto 1 & c & \mapsto 1 \\
e & \mapsto 2 & f & \mapsto 2
\end{align*}
\]
Let $\gamma$ denote the simple graph automorphism of $\Gamma_A$ defined above, which permutes $c$ and $d$, and let $\beta = \Psi^{-1}\gamma\Psi$. Then $\beta \in \text{Simp}(\sigma_3)$, and acts for example like

$$
\ldots 112002.02120011\ldots
\Psi \downarrow
\ldots bcfadf.dfcfaab\ldots
\gamma \downarrow
\ldots bdfacf.cfdfaab\ldots
\Psi^{-1} \downarrow
\ldots 102012.1202001\ldots
$$

Notice: $\beta$ essentially scans a string of 0, 1, 2's, and swaps 12 with 02!

Simp($\sigma_A$) is an important subgroup of Aut($\sigma_A$), and we’ll come back to it later.

1.2. **Representations of** Aut($\sigma$). Understanding intrinsic algebraic structure of Aut($\sigma$) is not easy. One of the few fundamental results about the algebraic structure of Aut($\sigma$) is the following.

**Theorem 1.3** (Ryan, [12]). If $A$ is primitive, the center of Aut($\sigma_A$) is generated by $\sigma_A$.

So how can we study Aut($\sigma$)? One way is to try to find good representations of it. There are two main classes of representations that we know of:

1. Periodic point representations, and representations derived from these.
2. The dimension representation.

The first, the periodic point representations (and ones derived from them), are quite natural to consider. They also lead to the gyration maps, which are also quite natural (once defined). The second, the dimension representation, is essentially a linear representation, and is based on the dimension group associated to the shift of finite type in question.

We start with the second one, the dimension representation.

1.3. **Dimension Representation.** We briefly recall the definition, introduced earlier by Mike, of the dimension group associated to a $\mathbb{Z}_+$-matrix. Given an $r \times r$ matrix $A$
over \( \mathbb{Z}_+ \) the eventual range subspace of \( A \) is \( ER(A) = \mathbb{Q}^r A^r \) (we will have matrices act on row vectors throughout), and the dimension group associated to \( A \) is

\[
\mathcal{G}_A = \{ x \in ER(A) \mid xA^k \in \mathbb{Z}^r \cap ER(A) \text{ for some } k \geq 0 \}.
\]

Recall also the group \( \mathcal{G}_A \) comes equipped with an automorphism (of abelian groups) \( \delta_A : \mathcal{G}_A \rightarrow \mathcal{G}_A \).

When \( A = (n) \) (the case of the full-shift on \( n \) symbols), the dimension group can be naturally identified with \( \mathbb{Z}^{[1/n]} \) where \( \delta_n(x) = xn \).

It is convenient to consider both the dimension group \( \mathcal{G}_A \) and its respective automorphism \( \delta_A \) as a pair \((\mathcal{G}_A, \delta_A)\). This is equivalent to considering \( \mathcal{G}_A \) not only as an abelian group, but also as a \( \mathbb{Z}[t] \)-module, where \( t \) acts by \( \delta_A^{-1} \) (as Mike described). (Note equivalently, considering \( \mathcal{G}_A \) as a \( \mathbb{Z}[t, t^{-1}] \)-module, since \( t \) acts invertibly on \( \mathcal{G}_A \) by definition.

By an automorphism of \((\mathcal{G}_A, \delta_A)\) we mean a group automorphism \( \Psi : \mathcal{G}_A \rightarrow \mathcal{G}_A \) which satisfies \( \Psi \delta_A = \delta_A \Psi \). In other words, we consider automorphisms of \( \mathcal{G}_A \) as a \( \mathbb{Z}[t, t^{-1}] \)-module. Define \( \text{Aut}(\mathcal{G}_A) \) to be the group of automorphisms of the pair \((\mathcal{G}_A, \delta_A)\).

From what Mike covered, we know that if \((X_A, \sigma_A)\) and \((X_B, \sigma_B)\) are topologically conjugate, then \( A \) and \( B \) are shift equivalent, and hence there is an isomorphism \((\mathcal{G}_A, \delta_A) \xrightarrow{\cong} (\mathcal{G}_B, \delta_B)\). It follows that any automorphism \( \alpha \in \text{Aut}(\sigma_A) \) induces an isomorphism \( \alpha_* : (\mathcal{G}_A, \delta_A) \xrightarrow{\cong} (\mathcal{G}_A, \delta_A) \). Thus there is a well-defined homomorphism

\[
(1.4) \quad \pi_A : \text{Aut}(\sigma_A) \rightarrow \text{Aut}(\mathcal{G}_A)
\]

which is known as the dimension representation of \( \text{Aut}(\sigma_A) \).

**Remark:** When \( A \) is over \( \mathbb{Z}_+ \), \( \mathcal{G}_A \) inherits an order structure making it into an ordered abelian group. We'll not consider this extra piece of data, and just consider \( \mathcal{G}_A \) as an abelian group (or \( \mathbb{Z}[t, t^{-1}] \)-module).

It is possible to define \( \mathcal{G}_A \) and \( \pi_A \) intrinsically using a construction of Krieger. However, we'll avoid doing this, and stick with the the strong shift equivalence view. In this point of view, one can explicitly compute the image of an automorphism under the dimension representation as follows. Given \( \alpha \in \text{Aut}(\sigma_A) \), \( \alpha \) is induced by a strong shift equivalence from \( A \) to \( A \)

\[
A = R_1S_1, A_2 = S_1R_1, \ldots, A_n = R_nS_n, A = S_nR_n.
\]
Then

$$\pi_A(\alpha) = \prod_{i=1}^{n} R_i \in \text{Aut}(G_A).$$

**Example 1.5.** The automorphism $\sigma_A \in \text{Aut}(\sigma_A)$ corresponds to the strong shift equivalence

$$A = (A)(I), \quad A = (I)(A).$$

In particular, we have for any shift of finite type $(X_A, \sigma_A)$

$$\pi_A(\sigma_A) = \delta_A \in \text{Aut}(G_A).$$

**Example 1.6.** When $A = (3)$, $ER(A) = \mathbb{Q}$, and as mentioned above, the dimension pair is $(\mathbb{Z}[\frac{1}{3}], \delta_3)$ where $\delta_3(x) = 3x$. Thus $\text{Aut}(G_3) \cong \mathbb{Z} \times \mathbb{Z}/2$, where the $\mathbb{Z}$ is generated by $\delta_3$, and the $\mathbb{Z}/2$ is generated by the map $x \mapsto -x$. The dimension representation then looks like

$$\pi_3: \text{Aut}(\sigma_3) \to \text{Aut}(\mathbb{Z}[\frac{1}{3}], \delta_3) \cong \mathbb{Z} \times \mathbb{Z}/2 = \langle \delta_3 \rangle \times \mathbb{Z}/2$$

$$\pi_3: \sigma_3 \mapsto \delta_3.$$

More generally, for full shifts the dimension representation takes on the following form.

**Proposition 1.7.** If $n$ has $j$ prime divisors, then $\text{Aut}(G_n) \cong \mathbb{Z}^j \times \mathbb{Z}/2$ and the map $\pi_A: \text{Aut}(\sigma_n) \to \text{Aut}(G_n)$ surjects onto the free abelian part of the $\text{Aut}(G_n)$.

**Proof.** From above we know $G_n \cong \mathbb{Z}[\frac{1}{n}]$. The $\mathbb{Z}/2$ component of $\text{Aut}(G_n)$ is generated by $x \mapsto -x$. The remaining free abelian component has basis given by the maps $\delta_{p_i}: x \mapsto x \cdot p_i$ where $p_i$ is a prime dividing $n$. For the second part, see [2].

In general, the dimension representation may not be surjective (see [8]), and the following question is still open:

**Q:** Given a shift of finite type $(X_A, \sigma_A)$, what is the range of the dimension representation $\pi_A: \text{Aut}(\sigma_A) \to \text{Aut}(G_A)$?

An automorphism $\alpha \in \text{Aut}(\sigma_A)$ is called inert if $\alpha$ lies in the kernel of $\pi_A$, and we denote the subgroup of inerts by

$$\text{Inert}(\sigma_A) = \ker \pi_A.$$

The subgroup $\text{Inert}(\sigma_A)$ is, roughly speaking, the heart of $\text{Aut}(\sigma_A)$, and in general, we do not know how to distinguish the subgroup of inert automorphisms among different shifts of finite type. The following is not hard to verify, and gives an indication that $\text{Inert}(\sigma_A)$ is large.
Proposition 1.8. For any shift of finite type \((X, \sigma_A)\), we have \(\text{Simp}(\sigma_A) \subset \text{Inert}(\sigma_A)\).

1.4. Periodic point representation. For an SFT \((X, \sigma)\) and \(k \in \mathbb{N}\) we let \(P_k\) denote the points of least period \(k\), and \(Q_k\) the set of orbits of length \(k\). For a shift of finite type, the set \(P_k\) is always finite, and we have

\[ |P_k| = k|Q_k|. \]

Let \(\alpha \in \text{Aut}(\sigma_A)\) and let \(k \in \mathbb{N}\). Since \(\alpha\) commutes with \(\sigma_A\), \(\alpha\) maps \(P_k\) to itself, and thus induces a permutation of \(P_k\) which we’ll denote by

\[ \rho_k(\alpha) \in \text{Sym}(P_k) \]

where \(\text{Sym}(P)\) of a set \(P\) denotes the group of permutations of \(P\).

It is to check that this assignment \(\alpha \mapsto \rho_k(\alpha)\) defines a homomorphism

\[ \rho_k : \text{Aut}(\sigma_A) \to \text{Sym}(P_k). \]

The automorphism \(\alpha\) must also respect \(\sigma_A\)-orbits, and it follows that \(\alpha\) induces a permutation of the set \(Q_k\) which we denote

\[ \xi_k(\alpha) \in \text{Sym}(Q_k). \]

Thus, we also get a homomorphism

\[ \xi : \text{Aut}(\sigma_A) \to \text{Sym}(Q_k). \]

The homomorphisms assemble into homomorphisms

\[ \rho : \text{Aut}(\sigma_A) \to \prod_{k=1}^{\infty} \text{Sym}(P_k) \]

(1.9)

\[ \rho(\alpha) = (\rho_1(\alpha), \rho_2(\alpha), \ldots). \]

and

\[ \xi : \text{Aut}(\sigma_A) \to \prod_{k=1}^{\infty} \text{Sym}(Q_k) \]

(1.10)

\[ \xi(\alpha) = (\xi_1(\alpha), \xi_2(\alpha), \ldots). \]

The map \(\rho\) is called the periodic point representation of \(\text{Aut}(\sigma_A)\), and \(\xi\) is called the orbit representation.

Remark: When \(A\) is primitive, the map \(\rho\) is in fact injective (this can be seen from the fact that for primitive \(A\), periodic points are dense in \((X, \sigma_A)\) - see [10]). This proves the following.

Proposition 1.11. Let \((X, \sigma_A)\) be a shift of finite type. Then \(\text{Aut}(\sigma_A)\) is residually finite.
The fact that $\text{Aut}(\sigma)$ is residually finite gives information about possible subgroups of $\text{Aut}(\sigma)$. For example, since any subgroup of a residually finite group is residually finite, this allows certain countable groups to be ruled out from embedding into $\text{Aut}(\sigma)$. For a concrete example, this implies the additive group of rationals $\mathbb{Q}$ cannot embed into $\text{Aut}(\sigma)$.

Fix $k \in \mathbb{N}$ and $\alpha \in \text{Aut}(\sigma_A)$. How can one determine $\rho_k(\alpha)$ from $\xi_k(\alpha)$? Since $\rho_k(\alpha)$ commutes with $\rho_k(\sigma_A)$, once one knows $\xi_k(\alpha)$, one need only know, for each orbit class, how much $\alpha$ ‘acts by $\sigma_A$’ on the orbit. Making this precise leads to considering the gyration maps, which were introduced by Boyle and Krieger in [1], and are defined as follows.

**Definition 1.12.** Fix $k \in \mathbb{N}$, $\alpha \in \text{Aut}(\sigma_A)$. We will define the $k$th gyration map $g_k : \text{Aut}(\sigma_A) \to \mathbb{Z}/k$ as follows. Let $Q_k = \{O_1, \ldots, O_{i(k)}\}$ denote the set of orbits in $Q_k$, and choose, for each $1 \leq i \leq k$, some representative point $x_i \in O_i$. Then $\alpha(x_i) \in O_{\xi_k(\alpha)(i)}$, so there exists some $r(\alpha, i) \in \mathbb{Z}/k$ such that $\alpha(x_i) = \sigma_{n(\alpha, i)}^r(x_{\xi_k(\alpha)(i)})$. Now define

$$g_k = \sum_{i=1}^{i(k)} r(\alpha, i) \in \mathbb{Z}/k.$$ 

Boyle and Krieger showed this map is independent of the choices of $x_i$’s, and is a homomorphism, so we get homomorphisms

$$g_k : \text{Aut}(\sigma_n) \to \mathbb{Z}/k.$$ 

Now we can define the *gyration representation* by

$$g : \text{Aut}(\sigma_n) \to \prod_{k=1}^{\infty} \mathbb{Z}/k$$

$$g(\alpha) = (g_1(\alpha), g_2(\alpha), \ldots)$$

**Remark:** The gyration maps are very natural. Restricting $\sigma_A$ to $P_k$ gives a finite dynamical system, and we can consider the group of automorphisms of this dynamical system, denoted $\text{Aut}(\sigma_A, P_k)$. Consider $\text{sign}\xi_k : \text{Aut}(\sigma_A, P_k) \to \mathbb{Z}/2$, the map $\xi_k$ composed with the sign map to $\mathbb{Z}/2$. It turns out, assuming there are at least 5 orbits, any other map from $\text{Aut}(\sigma_A, P_k)$ to an abelian group factors through the map $g_k \times \text{sign}\xi_k : \text{Aut}(\sigma_A, P_k) \to \mathbb{Z}/k \times \mathbb{Z}/2$.

1.5. **Inerts and the sign gyration compatibility condition.** Remarkably, the dimension representation and the gyration representation are not unconnected. It turns out that, at least for inert automorphisms, there are conditions which relate the periodic
orbit representation and the periodic point representation of the automorphism. This is formalized in the following way.

**Definition 1.14.** Say \( \alpha \in \text{Aut}(\sigma_A) \) satisfies **SGCC (sign-gyration compatibility condition)** if the following holds: for every positive integer \( m \) and every non-negative integer \( i \), if \( n = m2^i \), then

\[
g_n(\alpha) = 0 \text{ if } \prod_{j=0}^{i-1} \text{sign} \xi_{m2^j}(\alpha) = 1
\]

\[
g_n(\alpha) = \frac{n}{2} \text{ if } \prod_{j=0}^{i-1} \text{sign} \xi_{m2^j}(\alpha) = -1.
\]

The empty product we take to have the value 1.

Thus for \( \alpha \in \text{Aut}(\sigma_A) \) satisfying SGCC, \( g(\alpha) \) and \( \text{sign}\xi(\alpha) \) determine each other.

So which automorphisms satisfy SGCC? Amazingly enough, most of them.

**Theorem 1.16.** Let \((X_A, \sigma_A)\) be a shift of finite type. An automorphism \( \alpha \in \text{Aut}(\sigma_A) \) satisfies SGCC if:

1. (Boyle-Krieger in [1]) \( \alpha \) is a product of involutions (not for all \( A \)'s, but many, including the full shift).
2. (Nasu [11]) \( \alpha \) is a simple automorphism.
3. (Kim-Roush, with Wagoner, in [4]) \( \alpha \) is inert.

**Remark:** The SGCC conditions also give sufficient conditions for lifting an automorphism of a finite subsystem of a shift of finite type \((X_A, \sigma_A)\) (for \( A \) primitive) to \( \text{Aut}(\sigma_A) \). This is an important topic which we won’t discuss - see [9].

Finally, we end this lecture by mentioning two important conjectures in the study of \( \text{Aut}(\sigma_A) \):

**Simple finite order generation conjecture (SFOG):** For any shift of finite type \((X_A, \sigma_A)\), there is an equality

\[\text{Inert}(\sigma_A) = \text{Simp}(\sigma_A).\]

**Finite order generation conjecture (FOG):** For any shift of finite type \((X_A, \sigma_A)\), \( \text{Inert}(\sigma_A) \) is generated by elements of finite order.

Note that any simple automorphism is finite order, and hence SFOG implies FOG.
Similar to Williams’ Conjecture, both SFOG and FOG are known to be false for certain examples (that is to say, there exist shifts of finite type for which FOG (and hence SFOG) does not hold); see [5] for the SFOG example, and see [9] for the FOG examples. The counterexamples to FOG rely on a very difficult construction of Kim-Roush-Wagoner in [9], in which the polynomial matrix methods played an invaluable role (we do not know how to do such constructions without the polynomial matrix framework).

However, whether FOG or even SFOG might hold in the case of a full shift \((X_n, \sigma_n)\) is still unknown!

2. Lecture 8 - Wagoner’s SSE complexes and some applications

In the late 80’s, Wagoner introduced certain CW complexes to study strong shift equivalence. These CW complexes provide an algebraic topological/combinatorial framework for studying strong shift equivalence, and have played a key role in a number of important results in the study of shifts of finite type. Among these, one of the most important was the construction of counterexamples to Williams’ Conjecture in the primitive case, which were found by Kim and Roush in [7], and independently by Wagoner in [15]. Both the Kim and Roush strategy, and Wagoner’s strategy, take place in the setting of Wagoner’s strong shift equivalence complexes.

The goal in this last lecture is to give a brief introduction into these complexes. After defining and discussing them, we’ll give a short introduction into how the Kim-Roush and Wagoner strategies for producing counterexamples work. This will be very much an overview, and we will not go into details. We’ll spend slightly more time discussing Wagoner’s construction of an obstruction map, in part because of its connection with algebraic K-theory, which fits with our overall theme.

So in summary, the goal of this lecture is not to describe the construction of counterexamples to Williams’ Conjecture in any detail, but instead to give an overview and outline of how Wagoner’s spaces are built, how the counterexample strategies make use of them, and where they leave the state of the classification problem.

2.1. Wagoner’s SSE complexes. Suppose we have matrices \(A, B\) over \(\mathbb{Z}_+\), and two SSE’s from \(A\) to \(B\). We’ll draw these two SSE’s as two paths of ESSE’s, i.e. two chains of ESSE’s:
where each arrow in this picture represents an elementary strong shift equivalence. As we know from Mike’s lectures, each of these paths induce conjugacies

\[ \Psi_1: (X_A, \sigma_A) \rightarrow (X_B, \sigma_B) \]

and one may ask: when do two paths induce the same conjugacy? Can we determine this from the matrix entries in the paths themselves? One of the key insights in Wagoner’s Complexes is determining the correct relations on matrices to accomplish this. These relations are known as the Triangle Identities. Since the Triangle Identities lead directly to the definition of Wagoner’s Complexes, we’ll define both simultaneously.

**Definition 2.1.** Let \( R \) be a semi-ring. We define a CW-complex \( SSE(R) \) as follows:

1. The 0-cells of \( SSE(R) \) are square matrices over \( R \).
2. An edge from vertex \( A \) to vertex \( B \) corresponds to an elementary strong shift equivalence over \( R \) from \( A \) to \( B \):

\[
\begin{array}{c}
\begin{array}{c}
\text{A} \\
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
\text{B} \\
\end{array}
\end{array}
\]

\[
\begin{array}{c}
\begin{array}{c}
(\mathcal{R}, S) \\
\end{array}
\end{array}
\]

where \( A = RS, B = SR \).
3. 2-cells are given by triangles
which satisfy the *Triangle Identities*:

\[ R_1 R_2 = R_3, \quad R_2 S_3 = S_1, \quad S_3 R_1 = S_2. \]

The definition of \( SSE(\mathcal{R}) \) makes sense for any semi-ring. For this lecture however, we will only consider the case where \( \mathcal{R} \) is either:

1. \( \mathbb{Z}_0 = \{0, 1\} \)
2. \( \mathbb{Z}_+ \)
3. \( \mathbb{Z} \).

**Remark:** Wagoner also defines \( n \)-cells in \( SSE(\mathcal{R}) \) for \( n \geq 3 \) in [13], but we won’t need these in this lecture.

From the perspective of homotopy theory, the Triangle Identities dictate basic moves for paths in \( SSE(\mathcal{R}) \) to be homotopic. So why the Triangle Identities? In [13], Wagoner proved the following:

**Theorem 2.2.**

1. Two paths in \( SSE(\mathbb{Z}_0) \) induce the same conjugacy if and only if they are homotopic.
2. Two paths in \( SSE(\mathbb{Z}_+) \) induce the same conjugacy modulo simple automorphisms if and only if they are homotopic.

**Remark:** (2) is perhaps expected; recall the construction Mike outlined for associating conjugacies with \( SSE \)'s over \( \mathbb{Z}_+ \) requires a choice of labels for certain edges. This choice is precisely where ambiguity up to conjugating by simple automorphisms may arise.

Note that this gives (except for the third item) the following:

**Theorem 2.3** (Wagoner - see [14]). Let \( A \) be a square matrix over \( \mathbb{Z}_0 \). Then:

1. \( \text{Aut}(\sigma_A) \cong \pi_1(SSE(\mathbb{Z}_0), A) \).
2. \( \text{Aut}(\sigma_A)/\text{Simp}(\sigma_A) \cong \pi_1(SSE(\mathbb{Z}_+), A) \).
(3) \( \text{Aut}(G_A) \cong \pi_1(SSE(Z), A) \).

Upon using the identifications above, the composition map
\[
\pi_1(SSE(ZO), A) \to \pi_1(SSE(Z_+, A) \to \pi_1(SSE(Z), A)
\]
induced by the natural inclusions \( SSE(ZO) \hookrightarrow SSE(Z_+) \hookrightarrow SSE(Z) \) coincides with the dimension representation factoring as
\[
\text{Aut}(\sigma_A) \to \text{Aut}(\sigma_A)/\text{Simp}(\sigma_A) \to \text{Aut}(G_A).
\]

Wagoner also proves that \( \pi_k(SSE(ZO), A) = 0 \) for \( k \geq 2 \). This implies \( SSE(ZO), A \) is a model for the classifying space of \( \text{Aut}(\sigma_A) \), i.e. \( SSE(ZO), A \) is homotopy equivalent to \( B\text{Aut}(\sigma_A) \). Thus, for example, we have
\[
\text{Aut}(\sigma_A)_{ab} \cong H_1(\text{Aut}(\sigma_A), Z) \cong H_1((SSE(ZO), A), Z)
\]
and we remark that, we do not know what the abelianization \( \text{Aut}(\sigma_A)_{ab} \) is for any non-trivial shift of finite type \((X_A, \sigma_A)\). (Note: we do know the abelianization is not finitely-generated at least - see [2].)

2.2. Counterexamples to Williams’ Conjecture. Wagoner’s complexes provide a very useful framework for studying strong shift equivalence, and there have been a number of serious applications which use them. As mentioned earlier, one of these is constructing counterexamples to Williams’ Conjecture (counterexamples in the non-primitive case were produced by Kim and Roush in [6]; however, the more important case is the primitive case). In the primitive case, counterexamples were produced by Kim and Roush in [7], and independently by Wagoner in [15]. The technique for producing these counterexamples takes place in the setting of Wagoner’s SSE complexes. Both techniques build on a great deal of work by many authors; see [14] for a nice survey.

Our goal is to give a very brief introduction into how these counterexamples came to be. In fact we will not focus on the explicit counterexamples (which can be found in [7] and [15]), but instead on the strategy used to prove that they in fact are counterexamples.

First, both strategies roughly follow the same initial idea. Consider \( SSE(Z_+) \) as a subcomplex of \( SSE(Z) \), and, upon fixing a base point \( A \), consider the long exact sequence in homotopy groups for the pair \( (SSE(Z), SSE(Z_+)) \):
\[
\cdots \pi_1(SSE(Z), A) \to \pi_1(SSE(Z), SSE(Z_+, A) \to \pi_0(SSE(Z_+, A) \to \pi_0(SSE(Z, A)).
\]
(Note: the last three in this sequence are not actually groups, but just pointed sets. Still, exactness makes sense, by defining the kernel to be the pre-image of the base
point.)

The goal is to find \( A \) and \( B \) and show there is a path between them in \( SSE(\mathbb{Z}) \), but not a path between them in \( SSE(\mathbb{Z}_+) \). One can accomplish this by finding a function

\[
F : \pi_1(SSE(\mathbb{Z}), SSE(\mathbb{Z}_+), A) \to G
\]

to some \( G \) which vanishes on the image of \( \pi_1(SSE(\mathbb{Z}), A) \), but is not everywhere zero. Note, from earlier, we know \( \pi_1(SSE(\mathbb{Z}), A) \cong \text{Aut}(G_A) \), so being able to compute generators for \( \text{Aut}(G_A) \) plays an important role here.

Put another way, we want to find some group \( G \), and some function \( F \) from edges in \( SSE(\mathbb{Z}) \) to \( G \) which satisfies:

\[
(2.4) \quad F(\alpha * \beta) = F(\alpha) + F(\beta)
\]

\[
(2.5) \quad \text{If } \gamma_1 \text{ and } \gamma_2 \text{ are homotopic paths, then } F(\gamma_1) = F(\gamma_2)
\]

\[
(2.6) \quad F(\gamma) = 0 \text{ if } \gamma \text{ lies in } SSE(\mathbb{Z}_+)
\]

\[
(2.7) \quad F(\gamma) = 0 \text{ if } \gamma \in \pi_1(SSE(\mathbb{Z}), A).
\]

Kim and Roush, and then Wagoner, found such an \( F \) for \( G = \mathbb{Z}/2 \), in the case where \( A \) satisfies \( \text{tr}(A) = \text{tr}(A^2) = 0 \). Here is a rough sketch of how their ideas go:

**Kim-Roush relative sign-gyration method:** This method appeared first, and produced a counterexample in [7]. The method builds on the sign-gyration techniques for automorphisms outlined in the last lecture. The idea is to consider first the homomorphism

\[
sgcc_2 : \text{Aut}(\sigma) \to \mathbb{Z}/2
\]

\[
sgcc_2 = g_2 + \text{sign} \xi_1
\]

The map \( sgcc_2 \) is designed to check whether \( \alpha \) satisfies SGCC (defined last lecture) at least up to level two: that is, \( \alpha \) satisfies SGCC up period two points if and only if \( sgcc_2(\alpha) = 0 \). There are higher \( sgcc_k \)'s - see [8]), but for this discussion (and the construction of the counterexamples) it suffices to consider just \( sgcc_2 \).

Now what Kim and Roush did is the following:

1. Consider an ESSE \( A = RS, B = SR \), and let \( \Psi_{R,S} : (X_A, \sigma_A) \to (X_B, \sigma_B) \) be the corresponding conjugacy. Upon choosing a certain lexicographical ordering
of fixed points and period two points for each of $X_A, X_B$, one can compute $sgcc_2(\Psi_{R,S})$ w.r.t this ordering.

(2) This $sgcc_2(\Psi_{R,S})$ can be written as a formula just in terms of entries from $R$ and $S$, and this formula makes sense even if $R$ and $S$ are just over $\mathbb{Z}$!

Then Kim and Roush show that $F = sgcc_2$ and $G = \mathbb{Z}/2$ satisfies the three conditions above!

**Remark:** Adding a bit more, Kim, Roush and Wagoner in [8] actually showed that the map $sgcc_2$ : $\mathbb{Z}/2$ (in fact, $sgcc_k$) factors through the dimension representation.

**Wagoner’s $K_2$-valued obstruction map:** Wagoner, influenced by ideas from pseudo-isotopy theory, constructed such an $F$ landing in the $K$-theory group $K_2(\mathbb{Z}[t]/(t^2))$.

We will discuss in slightly more detail Wagoner’s construction. While this method is also somewhat technical, we choose it for a few main reasons:

(1) Landing in $K_2$, it fits in with the algebraic K-theory viewpoint we have been promoting.
(2) It operates within the polynomial matrix framework.
(3) It is perhaps suggestive of more general strategies for studying the refinement of strong shift equivalence over a ring by strong shift equivalence over the ordered part of a ring, i.e. (part (3) of Williams’ Problem in my earlier picture).

Before jumping in a bit more to Wagoner’s construction, let us make an important remark. Both the Kim-Roush method and Wagoner’s method can (at least in their current state) only produce counterexamples to Williams’ Conjecture by relying on the non-existence of periodic points at certain low levels. This is an important point which I’ll return to later.

**Remark:** While the two methods, one by Kim-Roush, and one by Wagoner, were independently developed to produce counterexamples, it was later shown that these methods, while differing greatly in their construction, are effectively the same (see the appendix in [15]).

**(Rough) sketch of Wagoner’s construction:** So how does Wagoner’s construction work? We will avoid defining $K_2$ of a ring $\mathcal{R}$ (see either [16, Chapter III], or forthcoming Appendix) and instead note two key facts we need about $K_2$:

(1) $K_2(\mathcal{R})$ is an abelian group.
(2) An expression of the form $\prod_{i=1}^k E_i = 1$, where $E_i$ are elementary matrices over $\mathcal{R}$, can be used to construct an element of $K_2(\mathcal{R})$. 
More information on $K_2(\mathcal{R})$ in general can be found in [16, Chapter III].

Given this, Wagoner’s construction proceeds as follows:

1. Consider an edge in $SSE(\mathbb{Z})$ from $A$ to $B$. As Mike showed in his lectures, this gives elementary $E_1, F_1$ over $\mathbb{Z}[t]$ such that
   \[ E_1(I - tA)F_1 = I - tB. \]

2. Suppose now $tr(A) = tr(A^2) = tr(B) = tr(B^2) = 0$. Then there exists elementary $E_2, F_2, E_3, F_3$ over $\mathbb{Z}[t]$ and $A', B'$ over $\mathbb{Z}[t]$ such that
   \[ E_2(I - tA)F_2 = I - t^2A', \]
   \[ E_3(I - tB)F_3 = I - t^2B'. \]

3. Thus we have, for some elementaries $X, Y$ over $\mathbb{Z}[t]$
   \[ X(I - t^2A')Y = I - t^2B'. \]

Passing to $\mathbb{Z}[t]/(t^2)$, we get
   \[ XY = I. \]

We can now use this expression to produce an element of $K_2(\mathbb{Z}[t]/(t^2))$.

Wagoner then showed:

1. This element of $K_2(\mathbb{Z}[t]/(t^2))$ depends only on the homotopy class of the path chosen.
2. This assignment is additive for composition of paths.
3. If the path lies entirely in $\mathbb{Z}_+$, then the corresponding element in $K_2(\mathbb{Z}[t]/(t^2))$ vanishes.

Finally, let us make an important remark. Without vanishing trace conditions, Step (2) above can not be carried out. Moreover, step (3) relies on the vanishing trace conditions. As a result, Wagoner’s construction is only defined in the case of shifts of finite type lacking periodic points of certain low order levels.

For the Kim-Roush technique the non-existence of low-order periodic points come in when they want to conclude that their assignment from edges to some $\mathbb{Z}/2$ vanishes along any path through $SSE(\mathbb{Z}_+)$; they do this by noting that, in $\mathbb{Z}_+$, their assignment coincides with the relative sign-gyration numbers associated to a conjugacy, which, in the absence of any periodic points of the given levels, must vanish.

Given these techniques for detecting counterexamples, it was then a matter of actually producing examples. From what I understand, this took some patience, cleverness, and some luck.
To finish, we highlight two open questions, one of which (Question 2) Mike already mentioned:

**Q1:** If $A$ is shift equivalent over $\mathbb{Z}_+$ to $(n)$, must $A$ be strong shift equivalent over $\mathbb{Z}_+$ to $(n)$? In other words, does Williams’ Conjecture hold in the case of full shifts?

**Q2:** Given $A$, is the refinement of the SE-$\mathbb{Z}_+$-equivalence class of $A$ by SSE-$\mathbb{Z}_+$ finite?

Finally, we think the complexes $SSE(\mathbb{Z})$ and $SSE(\mathbb{Z}_+)$ probably have much more to offer, and obtaining a deeper understanding of them would be valuable for studying both strong shift equivalence and the conjugacy problem for shifts of finite type.

**REFERENCES**


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