Simple Graphs and Zero-divisor Semigroups

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Abstract. In this paper, we provide examples of graphs which uniquely determine a zero-divisor semigroup. We show two classes of graphs that have no corresponding semigroups. Especially, we prove that no complete \( r \)-partite graph together with two or more end vertices (each linked to distinct vertices) has corresponding semigroups.

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1 Introduction

For any commutative semigroup \( S \) with zero element 0 (or any commutative ring \( S \)), there is an undirected zero-divisor graph \( \Gamma(S) \) associated with \( S \) (see [1–3, 7]). The vertex set of \( \Gamma(S) \) is the set of all non-zero zero-divisors of \( S \), and for distinct vertices \( x \) and \( y \) of \( \Gamma(S) \), there is an edge connecting \( x \) and \( y \) if and only if \( xy = 0 \). In [7] and [6], some fundamental properties and possible structures of \( \Gamma(S) \) were studied, where \( S \) is a semigroup. For example, for any semigroup \( S \), it was proved that \( \Gamma(S) \) is a connected simple graph with diameter less than or equal to 3, and the core of \( \Gamma(S) \) is a union of triangles and squares while any vertex of \( \Gamma(S) \) is either an end vertex or in the core if there exists a cycle in \( \Gamma(S) \). Many examples of graphs were given in [6, 7, 14] to give positive or negative answers to the following general problem: Given a connected simple graph \( G \), does there exist a semigroup \( S \) such that \( \Gamma(S) \cong G \)? The zero-divisor graphs were first extensively studied for commutative rings (see, e.g., [1–4]). For any semigroup \( S \), let \( T = Z(S) \) be the set of all zero-divisors of \( S \). Then \( T \) is an ideal of \( S \), and in particular, it is also a semigroup
with the property that all of its elements are zero-divisors of the semigroup \( T \). We call such semigroups zero-divisor semigroups. Obviously, we have \( \Gamma(S) \cong \Gamma(T) \). For a given connected simple graph \( G \), if there exists a zero-divisor semigroup \( S \) such that \( \Gamma(S) \cong G \), then we say that \( G \) has a corresponding semigroup and we call \( S \) a semigroup determined by the graph \( G \).

In this paper, we study semigroups determined by some graphs. We first give a class of graphs \( \Gamma_{\alpha} \) such that \( \Gamma_{\alpha} \) has a unique corresponding semigroup for each cardinal number \( \alpha \). The previous results in [6, 7, 14] show that most graphs have multiple corresponding semigroups. The number of semigroups corresponding to a graph increases rapidly if one end vertex is deleted. On the other hand, for a graph \( G \) having corresponding semigroups (e.g., the complete graph \( K_n \) together with an end vertex, or the complete bipartite graph \( K_{m,n} \) together with an end vertex), if we add more than two end vertices to \( G \), then the resulting graph may have no corresponding semigroups, as shown in the third section of this paper. This shows that the correspondence between semigroups and the possible graphs is rather sensitive.

All semigroups in this paper are multiplicatively commutative zero-divisor semigroups with zero element 0, where 0 \( = 0 \) for all \( x \in S \), and all graphs in this paper are undirected, simple, and connected. For any vertices \( x, y \) in a graph \( G \), if \( x \) and \( y \) are adjacent, we denote it as \( x - y \). Refer to [5] for other graph notations adopted in this paper and refer to [8] for concepts and results about semigroups.

2 Semigroups Uniquely Determined by Some Graphs

Example 2.1. For any finite or infinite set \( A \) with \( \alpha \) elements, there is an associated commutative semigroup \( P_{\alpha} = \{ x_B \mid B \subseteq A \} \). The multiplication of \( P_{\alpha} \) is defined by \( x_C \cdot x_B = x_{C \cap B} \). It is straightforward to verify that \( P_{\alpha} \) is a commutative semigroup with the identity element \( x_A \). Also, \( x_\emptyset \) is the zero element of \( P_{\alpha} \), i.e., \( x_\emptyset \cdot x_B = x_\emptyset \) for each element \( x_B \in P_{\alpha} \). Then we have semigroup isomorphisms \( P_{\alpha} \cong (2^A, \cap) \cong (2^A, \cup) \), where \( 2^A \) is the power set of \( A \). Let \( Z_2 \) be the ring of integers modulo 2. For any finite number \( n \), let \( Z_2^{(n)} \) be the ring direct sum of \( n \) copies of \( Z_2 \) and consider its multiplicative semigroup \( (Z_2^{(n)}, \cdot) \). If \( |A| = n \), it is easy to verify that the map

\[
\sigma : (2^A, \cap) \to (Z_2^{(n)}, \cdot), \quad B \mapsto (y_1, y_2, \ldots, y_n), \quad \text{where } y_i = \begin{cases} 0 & \text{if } i \notin B, \\ 1 & \text{if } i \in B, \end{cases}
\]

is a semigroup isomorphism.

Denote by \( \Gamma_{\alpha} \) the zero-divisor graph of \( P_{\alpha} \). \( \Gamma_{\alpha} \) is a symmetric graph with a moderate number of edges. We list some properties of \( \Gamma_{\alpha} \) for any finite \( \alpha = n \):

1. \( V(\Gamma_{n}) = P_n = \{0, 1\} \) and hence it contains \( |V(\Gamma_{n})| = 2^n - 2 \) vertices.

2. For any \( x_B \in \Gamma_{n} \) with \( |B| = i \), let \( N(x_B) \) be the neighborhood of \( x_B \), i.e., \( N(x_B) = \{ y \in \Gamma_{n} \mid x - y \in \Gamma_{n} \} \). Then \( |N(x_B)| = C^{1}_{n-i} + C^{2}_{n-i} + \cdots + C^{i}_{n-i} = 2^{n-i} - 1 \).

3. The edge number is \( |E(\Gamma_{n})| = C^{1}_{n}2^{n-2} + C^{2}_{n}2^{n-3} + \cdots + C^{n-1}_{n}2^{n-2} + 1 \).
4. The clique number of $\Gamma_n$ is $n$. When $n \geq 3$, the diameter of $\Gamma_n$ is 3 and $\Gamma_n$ has $n$ end vertices.

5. The automorphism group of $\Gamma_n$ is the symmetric group $S_n$. Thus, this graph is highly symmetric.

Especially, $\Gamma_2$ is just the complete graph $K_2$, and $\Gamma_3$ is the complete graph $K_3$ together with three end vertices linked with distinct vertices of $K_3$. The graph $\Gamma_4$ has 14 vertices and 25 edges, while $|V(\Gamma_5)| = 30$ and $|E(\Gamma_5)| = 90$.

For this graph $\Gamma_n$ with a moderate edge set, we have the following result:

**Theorem 2.2.** Assume $|A| = \alpha \geq 3$ and let $P_\alpha = \{x_B \mid B \subseteq A\}$ be the commutative semigroup defined on $2^A$ with graph $\Gamma_\alpha$. If $S$ is a commutative zero-divisor semigroup whose graph $\Gamma(S)$ is isomorphic to $\Gamma_\alpha$, then $S$ is isomorphic to the zero-divisor semigroup $P_\alpha = \{1\}$.

**Proof.** Assume that $S$ is a zero-divisor semigroup such that $\Gamma(S) = \Gamma_\alpha$. By the property of $\Gamma_\alpha$, we can have a labeling $x$ to the elements of $S^* = S \setminus \{0\}$

$$S = \{x_B \mid B \text{ is a proper subset of } A\}$$

such that $0 = x_\emptyset$, and for any distinct elements $x_B, x_C \in S$, $x_Bx_C = 0$ if and only if $B \cap C = \emptyset$.

(1) For any $d \in A$ and any $D \subset A$ with $\{d\} \subset D$, we can assume $x_dx_D = x_E$ for some non-empty $E$. We claim that $d \in E$. In fact, if $d \notin E$, we have distinct elements $r, s \neq d$ in $A$ such that $r \in E$. If $E = \{r\}$, then let $H = \{r, s\}$. If $|E| \geq 2$, then take $H = \{r\}$. In each case, $E \neq H$, $E \cap H = \{r\}$, and hence $x_Ex_H \neq 0$. On the other hand, we have $0 = x_Hx_dx_D = x_Hx_E$. This contradiction shows that whenever $\{d\} \subset D \subset A$, we have a proper subset $E$ of $A$ such that $x_dx_D = x_E$ and $d \in E$. If further there exists some $m \in E - \{d\}$, then $0 \neq x_Ex_m = x_dx_mx_D = 0x_D = 0$, a contradiction. Thus, we obtain $x_dx_D = x_d$.

(2) For any $d \in A$, we show that $x_d^2 = x_d$. First we claim that $x_dx_d = x_d$ holds for distinct elements $r, s, d \in A$. In fact, we can assume $x_dx_ds = x_C$ for some non-empty proper subset $C$ of $A$. If $d \notin C$, then $0 = x_dx_C = (x_dx_d)x_d = x_dx_ds = x_d$. So we must have $d \in C$. If $\{d\} \subset C$, then we take $m \in C - \{d\}$ and we have $x_m = x_Cx_m = x_d(x_dx_m)$. Since $r \neq s$, either $r \notin m$ or $s \notin m$. Either case gives $x_dx_d = m \neq 0$, a contradiction. Thus, $C = \{d\}$ and hence $x_d = x_d$.

Now by taking distinct elements $r, s \neq d$ in $A$, we have $x_d = x_dx_d = (x_dx_d)x_d = x_d^2$. By (1), we obtain $x_dx_D = x_d$ for all $d \in D \subset A$.

(3) In the following, we want to prove that $x_Bx_C = x_{B \cap C}$ holds for any proper subsets $B, C$ of $A$, and this will prove the uniqueness of the zero-divisor semigroup $S$ with $\Gamma(S) \cong \Gamma_\alpha$.

If $B \cap C = \emptyset$, then we are done.

Assume $B \neq C$ and $B \cap C \neq \emptyset$. Then $E \neq \emptyset$ in $x_Bx_C = x_E$. If $E \not\subseteq B \cap C$, then there is an element $d \in E - B \cap C$, and assume further that $d \notin B$. Then $x_d = x_dx_E = (x_dx_B)x_C = 0$, a contradiction. If $B \cap C \not\subseteq E$, then there exists an element $d \in B \cap C - E$. Then $0 = x_dx_E = (x_dx_B)x_C = x_d$, another contradiction. These contradictions show that $x_Bx_C = x_{B \cap C}$ whenever $B \neq C$ and $B \cap C \neq \emptyset$.
If \( B = C \neq \emptyset \), we obtain \( x_d(x_B x_C)x_d = (x_d x_B)(x_C x_d) = x_d^2 = x_d \) for any \( d \in B \), and therefore \( x_B^2 \neq 0 \). Assume \( x_B^2 = x_E \) with \( E \neq \emptyset \). If \( B \subseteq E \), then for any \( d \in B - E \), we have \( 0 = x_d x_E = x_d x_B x_B = x_d \), a contradiction. If \( E \subseteq B \), then for any \( r \in E - B \), we have \( 0 = x_r x_B^2 = x_r x_E = x_r \), another contradiction. The contradictions show that \( x_B^2 = x_B \) holds for any proper subset \( B \) of \( A \). This completes the proof.

**Remark.** It is known that \( (2^A, \wedge, \vee, A, \emptyset) \) is a Boolean algebra for any set \( A \). Conversely, each finite Boolean algebra is isomorphic to \( (2^A, \wedge, \vee, A, \emptyset) \) for some finite set \( A \). According to the famous Stone Representation Theorem, each infinite Boolean algebra is isomorphic to a subalgebra of \( (2^A, \wedge, \vee, A, \emptyset) \) for some infinite set \( A \). It is also known that Boolean algebras and Boolean rings are essentially the same algebraic system. Theorem 2.2 is recently generalized to arbitrary Boolean rings in [9, Theorem 4.2].

**Proposition 2.3.** The following graph \( G \) has a unique corresponding zero-divisor semigroup:

```
\begin{tikzpicture}
  \node (a) at (0,0) {a};
  \node (d) at (1,0) {d};
  \node (b) at (0,-1) {b};
  \node (e) at (1,-1) {e};
  \node (c) at (0.5,-1.5) {c};

  \draw (a) -- (d);
  \draw (a) -- (b);
  \draw (b) -- (c);
  \draw (c) -- (e);
\end{tikzpicture}
```

**Figure 1**

**Proof.** Since \( ac \in \text{ann}(d) \cap \text{ann}(b) \cap \text{ann}(e) \), we have \( ac = e \) and hence \( c^2 = 0 \). By symmetry, we also have \( dc = b \) and \( b^2 = 0 \). From \( ac \in \text{ann}(e) \cap \text{ann}(b) \cap \text{ann}(d) \), we obtain \( ae = e \). By symmetry, \( db = b \). From \( ac = e \) and \( dc = b \), we obtain \( c^2 a = 0 \) and \( c^2 d = 0 \). Thus, \( c^2 \in \text{ann}(e) \cap \text{ann}(b) \cap \text{ann}(a) \cap \text{ann}(d) \) and hence \( c^2 = 0 \). Finally, we consider possible values of \( a^2 \). First, \( a^2 \neq 0 \) by \( ae = e \). By the known equalities, we have \( a^2 \neq 0, b, e, c, d \). Thus, \( a^2 = a \). By symmetry, we also have \( d^2 = d \). It is routine to verify that the multiplication thus defined is associative. For reader’s convenience, we list the commutative multiplication table in the following:

<table>
<thead>
<tr>
<th></th>
<th>a</th>
<th>d</th>
<th>b</th>
<th>c</th>
<th>e</th>
</tr>
</thead>
<tbody>
<tr>
<td>a</td>
<td>a</td>
<td>0</td>
<td>0</td>
<td>e</td>
<td>e</td>
</tr>
<tr>
<td>d</td>
<td>d</td>
<td>b</td>
<td>b</td>
<td>0</td>
<td></td>
</tr>
<tr>
<td>b</td>
<td></td>
<td>d</td>
<td>b</td>
<td>b</td>
<td>0</td>
</tr>
<tr>
<td>c</td>
<td></td>
<td></td>
<td>0</td>
<td>0</td>
<td></td>
</tr>
<tr>
<td>e</td>
<td></td>
<td></td>
<td></td>
<td>0</td>
<td></td>
</tr>
</tbody>
</table>

This completes the proof.

**Remarks.** In [11, Proposition 3.1], a uniqueness result was also obtained for the directed zero-divisor graphs of non-commutative rings \( R \), i.e., for any ring \( R \), if \( \Gamma(R) \) has a source vertex (resp., a sink vertex) \( x \) such that \( x^2 = 0 \), then \( R \) is uniquely
determined, and is isomorphic to the following ring

\[
\left\{ \begin{pmatrix} a & 0 \\ b & 0 \end{pmatrix} \mid a, b \in \mathbb{Z}_2 \right\} \quad \text{(resp.,} \quad \left\{ \begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} \mid a, b \in \mathbb{Z}_2 \right\}.
\]

Furthermore, this condition was proved to be equivalent to the condition that the graph \( \Gamma(R) \) has exactly one source vertex (resp., sink vertex).

By a further discussion, if we add a finite or an infinite number of end vertices adjacent to either (or both) of the vertices \( e, b \) in Proposition 2.3, then the resulting graph still has a unique zero-divisor semigroup.

By [10, Theorem 2.2], the path graph \( P_4 \) with four vertices has a unique zero-divisor semigroup.

Let \( n \) be any finite or infinite cardinal number not less than 4. According to [13, Theorem 1], the graph \( K_n + 2 \) (i.e., the complete graph \( K_n \) together with two end vertices that adjacent to distinct vertices) has a unique corresponding zero-divisor semigroup \( S \) such that \( \Gamma(S) \cong K_n + 2 \). By [13, Theorem 2], the graph \( K_n + 3 \) has no corresponding zero-divisor semigroups.

By Theorem 2.2, the graph \( K_3 + 3 \) has a unique zero-divisor semigroup although \( K_n + r \) has no corresponding semigroups for any \( n \geq 4 \) and \( r \geq 3 \) (see Corollary 3.2 below). We know that \( K_n + 2 \) has a unique zero-divisor semigroup for any \( n \geq 4 \), while \( K_3 + 2 \) has multiple zero-divisor semigroups.

In the following, we determine all mutually non-isomorphic zero-divisor semigroups whose zero-divisor graph is isomorphic to \( K_3 + 2 \). This is the final unsolved case in studying the correspondence between the category of zero-divisor semigroups and the category of all complete graphs possibly together with end vertices.

**Proposition 2.4.** \( K_3 + 2 \) has three mutually non-isomorphic zero-divisor semigroups.

**Proof.** Consider the following graph \( K_3 + 2 \):

\[ u \quad \xrightarrow{a} \quad b \quad \xrightarrow{c} \quad v \]

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{Figure2.png}
\caption{Figure 2}
\end{figure}

Since \( ub \in \text{ann}(v) \cap \text{ann}(a) \), we have \( ub = b \). By symmetry, we also obtain \( va = a \). It follows that \( u^2 \neq 0 \) and \( v^2 \neq 0 \).

Now consider \( uv \). If \( uv = a \), then \( u^2v = ua = 0 \) and \( (cu)v = ca = 0 \). Therefore, \( u^2 = b = cu \). Then one has \( 0 = cb = (cu)u = bu \), a contradiction. Thus, \( uv \neq a \), and by symmetry, \( uv \neq b \). It follows that \( uv = c \).

Since \( u^2b = b \) and \( u^2v = uc \neq 0 \), we have \( u^2 = u \). By symmetry, we also obtain \( v^2 = v \). Thus, \( c^2 = c \).

Obviously, \( \{uc, vc\} \subseteq \{a, b, c\} \). Now consider \( uc \). If \( uc = a \), then \( uc = ua = 0 \), a contradiction. If \( uc = b \), then \( c^2 = (vu)c = vb = 0 \), a contradiction. This shows that \( uc = c \). By symmetry, we also obtain \( vc = c \).
Finally, consider the values of remaining squares. Since \(a^2u = 0\), we have \(a^2 = 0\) or \(a^2 = a\). Similarly, \(b^2 = 0\) or \(b^2 = b\).

By the above discussion, there are at most three mutually non-isomorphic zero-divisor semigroups \(S\) such that \(\Gamma(S) \cong K_3 + 2\). Below we list the three multiplication tables on \(S = \{a, b, c, u, v\}\) such that \(\Gamma(S) \cong K_3 + 2\), and we can verify the associativity in each case via a computer program:

\[
\begin{array}{c|ccc|ccc}
\cdot & a & b & c & u & v & \cdot \\
\hline
a & 0 & 0 & 0 & 0 & a & a \\
b & 0 & b & 0 & b & 0 & b \\
c & c & c & c & c & c & c \\
u & u & c & u & c & u & c \\
v & v & v & v & v & v \\
\end{array}
\]

This completes the proof.

\[ \square \]

Remark. In [12], we gave formulas to count the numbers of semigroups whose zero-divisor graph is \(K_n\) or \(K_n + 1\) for all \(n \geq 3\).

3 Two Classes of Graphs Which Have No Corresponding Semigroups

For convenience, we call a subset \(\{u, v\} \subseteq V(A)\) of a graph \(A\) a sub-center of \(A\) if \(u \neq v\) and each vertex in \(A\) is adjacent to either \(u\) or \(v\). For any \(v \in V(A)\), denote \(N(v) = N(v) \cup \{\{u\}\}\), where \(N(v) = \{u \in V(A) \mid u \neq v, u - v\}\) is the neighborhood of \(v\) in the graph \(A\).

**Theorem 3.1.** Let \(G\) be a simple connected finite or infinite graph such that \(V(G) = A \cup \left( \bigcup_{i=1}^m X_i \right)\) which is a disjoint union of \(m + 1\) non-empty subsets, where \(|A| \geq m \geq 4\) (\(m\) could be infinite). If there exist \(m\) distinct vertices \(a_i\) in \(A\) such that \(a_1, a_2\) is a sub-center of the subgraph \(A\) and, for each \(i, \bigcup_{x \in X_i} N(v) \subseteq X_i \cup \{a_i\}\), then \(G\) is not the zero-divisor graph of any semigroup \(S\).

**Proof.** Suppose on the contrary that there is a zero-divisor semigroup \(S\) such that \(\Gamma(S) \cong G\). Without loss of generality, we may assume \(S - \{0\} = V(G)\).

By assumption, there exists \(x_i \in X_i\) such that \(a_i x_i = 0\). For any \(1 \leq i, j \leq 4\) with \(i \neq j\), since \(x_i(a_j x_i) = x_i a_j x_i = 0\), we have \(a_j x_i \in N(x_j) \subseteq X_j \cup \{a_j\}\). Since \((a_j x_i) a_i = a_j (a_i x_i) = 0\), we obtain \(a_j x_i \notin X_j\) by assumption. Thus, \(a_j x_i = a_j\) for all \(1 \leq i, j \leq 4\) with \(i \neq j\).

Now for the above mentioned elements \(x_3 \in X_3\) and \(x_4 \in X_4\), consider \(x_3 x_4\). Set \(z = x_3 x_4\). Then \(z \neq 0\), and so either \(z \in A\) or \(z \in X = \bigcup_{i=1}^m X_i\). By the above discussion, we have \(a_1 z = a_1 x_3 x_4 = a_1 \neq 0\) and \(a_2 z = a_2 \neq 0\). Since \(\{a_1, a_2\}\) is a sub-center of the graph \(A\), it follows that \(z \notin A\). So \(z \in X\). But this contradicts the assumption and the fact that \(a_3 z = (a_3 x_3) x_4 = 0\) and \(a_4 z = 0\).

By Theorem 3.1 and its proof, we have the following:

**Corollary 3.2.** The following graphs have no corresponding semigroups:

1. \(M_{n,k} = K_n \cup \left( \bigcup_{i=1}^k \{x_i\} \right)\), i.e., the complete graph \(K_n = \{a_1, a_2, \ldots, a_n\}\) together with end vertices \(x_i\) such that \(a_i \sim x_i\) \((n \geq k \geq 4)\).
Any generalization of graphs with one of the forms:

![Diagram](Figure 3)

![Diagram](Figure 4)

By [6, Theorem 3(2)], any complete $r$-partite graph (resp., any complete $r$-partite graph together with an end vertex) is the graph of a semigroup. The complete $r$-partite graph case was also independently discovered in [14, Proposition 3.2]. Like the complete graph case, a step further can lead to a negative result.

**Theorem 3.3.** Let $G$ be a connected simple graph with $V(G) = A \cup B \cup X_1 \cup Y_1$ a disjoint union of four non-empty subsets, where $|A| \geq 2$ and $|B| \geq 2$. If $G$ satisfies the following conditions, then $G$ is not the zero-divisor graph of any semigroup:

1. There exist four elements $a_i \in A$ and $b_i \in B$ ($i = 1, 2$) such that $a_i$ (resp., $b_i$) is adjacent to each vertex in $B$ (resp., in $A$) for $i = 1, 2$.
2. $\bigcup_{i \in X_1} N(v) \subseteq X_1 \cup \{a_1\}$ and $\bigcup_{i \in Y_1} N(v) \subseteq Y_1 \cup \{b_1\}$.
3. The induced subgraph on $A \cup B$ is a subgraph of the complete bipartite graph $K_{A,B}$.

**Proof.** Assume on the contrary that there exists a commutative zero-divisor semigroup $S$ such that $\Gamma(S) \cong G$. First, similar to the proof of Theorem 3.1, it can be verified that $a_1y = a_1$ and $b_1x = b_1$ for all $x \in X_1$ and $y \in Y_1$. This implies especially that $x^2 \neq 0$ and $y^2 \neq 0$ for all $x \in X_1$ and $y \in Y_1$.

Now fix two elements $x \in X_1$ and $y \in Y_1$, and consider $xy$. If $xy = a_i$ ($i \neq 1$), then $a_1a_i = 0$, a contradiction. Similarly, we also have $xy \neq b_j$ for any $j \neq 1$. If $xy = x_1 \in X_1$, then $b_1x_1 = 0$, a contradiction. Similarly, we also have $xy \notin Y_1$. This proves that either $xy = a_1$ or $xy = b_1$. By symmetry, we can assume $xy = a_1$.

Then we obtain $a_1^2 = 0$ and $0 = a_1x = x^2y$. Since $y^2 \neq 0$, we have either $x^2 = b_1$ or $x^2 = y_1 \in Y_1$. But if $x^2 = y_1$, then $a_1y_1 = a_1x^2 = 0$, a contradiction. It follows that $x^2 = b_1$ and thus $b_1^2 = b_1x^2 = b_1$. From $xy = a_1$, we also obtain $0 = b_1a_1 = (b_1y)x$ ($r \geq 2$). Thus, we have $b_1y = z_r$, where either $z_r = a_1$ or $z_r = x_2 \in X_1$.

Finally, we consider $a_2y$. If $a_2y = a_r$ ($r \geq 1$), then by assumption, $0 = b_2a_r = (b_2y)a_2 = z_2a_2$, where either $z_2 = a_1$ or $z_2 = x_2 \in X_1$, a contradiction in each case. If $a_2y = b_1$, then $b_1^2 = 0$, contradicting $b_1^2 = b_1$. If $a_2y = b_j$ for some $j \geq 2$, then $b_1b_j = 0$, contradicting the assumption again. Hence, we must have $a_2y \notin X_1 \cup Y_1$. But if $a_2y = x_1 \in X_1$, then $b_1x_1 = 0$; and if $a_2y = y_1 \in Y_1$, then $b_2y_1 = 0$. In conclusion, $a_2y \not\in S$ and this completes the proof.

**Corollary 3.4.** For any $m \geq 2$ and $n \geq 2$, let $L_{m,n}$ be the complete bipartite graph $K_{m,n}$ together with at least two end vertices which connect to distinct vertices of
Then $L_{m,n}$ has no corresponding semigroups. In particular, the following graph has no corresponding semigroups:

```
  o   o   o
```

**Figure 5**

*Proof.* If at least two end vertices, say $x$ and $y$, connect to one part of $K_{m,n}$, then the distance from $x$ to $y$ is 4. Thus, this $L_{m,n}$ has no corresponding semigroups.

The other case is that $L_{m,n}$ has exactly two end vertices and they connect to different parts of $K_{m,n}$. In this case, the result follows directly from Theorem 3.3. □

We remark that the above corollary also holds for the complete $r$-partite graph case ($r \geq 2$) such that each partition has at least two elements.

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