A sharp upper bound on the largest eigenvalue of the Laplacian matrix of a graph

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Abstract

Let \( G \) be a simple connected graph with \( n \) vertices. The largest eigenvalue of the Laplacian matrix of \( G \) is denoted by \( \mu(G) \). Suppose the degree sequence of \( G \) is \( d_1 \geq d_2 \geq \cdots \geq d_n \). In this paper, we present a sharp upper bound of \( \mu(G) \)

\[
\mu(G) \leq d_n + \frac{1}{2} + \sqrt{\left( d_n - \frac{1}{2} \right)^2 + \sum_{i=1}^{n} d_i (d_i - d_n)},
\]

the equality holds if and only if \( G \) is a regular bipartite graph. © 2002 Published by Elsevier Science Inc.

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1. Introduction

Let \( G = (V, E) \) be a simple connected graph with vertex set \( V = \{v_1, v_2, \ldots, v_n\} \) and edge set \( E = \{e_1, e_2, \ldots, e_m\} \). We denote the line graph of \( G \) by \( L_G \). Let \( A(G) \) be the adjacency matrix of graph \( G \). Denoting the degree of \( v_i \in V(G) \) by \( d_i, d(v_i) \) or \( d_G(v_i) \), \( d_1 \geq d_2 \geq \cdots \geq d_n \). \( D = D(G) = \text{diag}\{d_1, d_2, \ldots, d_n\}, \) \( L = L(G) = \)

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$D(G) - A(G)$, $K = K(G) = D(G) + A(G)$. Then $L(G)$ is the Laplacian matrix of $G$. Let $Q = Q(G)$ be a vertex-edge incidence matrix of $G$. Thus

$$K(G) = D(G) + A(G) = QQ^T \quad \text{and} \quad Q^TQ = 2I_m + A(L_G).$$

The adjacency spectral radius, $\rho(G)$, of $G$ is the largest eigenvalue of $A(G)$. The Laplacian spectral radius, $\mu(G)$, of $G$ is the largest eigenvalue of $L(G)$. Let $\mu'$ be the largest eigenvalue of $K(G)$.

Merris [1] pointed out

$$\mu(G) \leq \mu' = 2 + \rho(L_G),$$

and the equality holds if $G$ is a bipartite graph. In this paper, we first prove that the equality holds if and only if $G$ is a bipartite graph.

There are many known upper bounds for $\mu(G)$.

In 1985, Anderson and Morley [2] showed that

$$\mu(G) \leq \max \{d(u) + d(v) | (u, v) \in E(G)\}.$$ 

In 1997, Li and Zhang [4] gave an upper bound of $\mu(G)$

$$\mu(G) \leq 2 + \sqrt{(r - 2)(s - 2)},$$

where $r = \max\{d(u) + d(v) | (u, v) \in E(G)\}$ and suppose $(x, y) \in E(G)$ satisfies $d(x) + d(y) = r$, $s = \max\{d(u) + d(v) | (u, v) \in E(G) - (x, y)\}$.

In 1998, Merris [3] gave that

$$\mu(G) \leq \max \{d(v) + m(v) | v \in V(G)\},$$

where $m(v)$ is the average of the degrees of the vertices adjacent to $v$. $d(v)m(v)$ is the “2-degree” of $v$.

In 1998, Li and Zhang [5] proved that

$$\mu(G) \leq \max \left\{ \frac{d(u)(d(u) + m(u)) + d(v)(d(v) + m(v))}{d(u) + d(v)} \right\}.$$ 

In this paper, using the degree sequence $(d_1, d_2, \ldots, d_n)$, we give a sharp upper bound

$$\mu(G) \leq d_n + \frac{1}{2} + \sqrt{\left(d_n - \frac{1}{2}\right)^2 + \sum_{i=1}^{n} d_i(d_i - d_n)},$$

the equality holds if and only if $G$ is a regular bipartite graph.

The terminology not defined here can be found in [7].

2. Lemmas and main results

**Lemma 1.** Let $B$ be a real symmetric nonnegative irreducible matrix and $\lambda$ be the largest eigenvalue of $B$. $Z \in \mathbb{R}^n$. If $Z^T B Z = \lambda$ and $\|Z\| = 1$, then $BZ = \lambda Z$. 


Proof. Since $B$ is a real symmetric matrix, we can denote its eigenvalues by $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$, $w_i$ is eigenvector corresponding to $\lambda_i$ $(i = 1, 2, \ldots, n)$ such that $W = (w_1, w_2, \ldots, w_n)$ is an orthogonal matrix and $\|W\| = 1$. Thus

$$W^tBW = W^{-1}BW = \text{diag}\{\lambda_1, \lambda_2, \ldots, \lambda_n\} = T,$$

$$B = WTW^{-1} = WTW^t.$$

Since $Z^tBZ = \lambda$, we have

$$\lambda = Z^tBZ = Z^tWTVZ = S^tTS = \sum_{i=1}^n \lambda_i s_i^2 \leq \sum_{i=1}^n \lambda_1 s_i^2 = \lambda_1 = \lambda,$$

where $S = (s_1, s_2, \ldots, s_n)^t = W^tZ$, $\|S\| = \|WTW^t\| = 1$.

Since $B$ is a nonnegative irreducible matrix, using the Perron–Frobenius’ theorem, we know $\lambda$ is a unique largest eigenvalue of $B$. So

$$s_2 = s_3 = \cdots = s_n = 0, \quad s_1 = \pm 1, \quad Z = WS = \pm w_1.$$

Therefore

$$BZ = \lambda Z,$$

i.e. $Z$ is an eigenvector of $B$ belonging to $\lambda$. □

Lemma 2. If $G$ is a connected graph, then

$$\mu(G) \leq 2 + \rho(L_G),$$

the equality holds if and only if $G$ is a bipartite graph.

Proof. Since $\mu' = 2 + \rho(L_G)$, we need only to prove $\mu \leq \mu'$.

Let $Y = (y_1, y_2, \ldots, y_n)^t \in \mathbb{R}^n$, $\|Y\| = 1$. Let $X = (x_1, x_2, \ldots, x_n)^t \in \mathbb{R}^n$ be a unit eigenvector of $L$ belonging to $\mu$ and $X' = (x'_1, x'_2, \ldots, x'_n)^t \in \mathbb{R}^n$ be a unit eigenvector of $K$ belonging to $\mu'$. Let $|X| = (|x_1|, |x_2|, \ldots, |x_n|)^t$.

(1) First, we prove that $\mu \leq \mu'$.

$$\mu = \max Y^tLY$$

$$= \max Y^t(D - A)Y$$

$$= \max \sum_{v_i \sim v_j, i < j} (y_i - y_j)^2$$

$$= X^t(D - A)X$$

$$= \sum_{v_i \sim v_j, i < j} (x_i - x_j)^2$$

and

$$\mu' = \max Y^tKY$$

$$= \max Y^t(D + A)Y$$
\[= \max_{v_i \sim v_j, \ i < j} (y_i + y_j)^2\]

\[= X^t (D + A) X'\]

\[= \sum_{v_i \sim v_j, \ i < j} (x'_i + x'_j)^2.\]

Thus

\[\mu = \sum_{v_i \sim v_j, \ i < j} (x_i - x_j)^2\]

\[\leq \sum_{v_i \sim v_j, \ i < j} (|x_i| + |x_j|)^2\]

\[= |X|^t (D + A) |X|\]

\[\leq \mu'. \quad (\ast)\]

(2) When \(\mu = \mu'\), all inequalities (\(\ast\)) must be equalities. By Lemma 1 and the equality

\[|X|^t (D + A) |X| = \mu',\]

we know \(|X|\) is an eigenvector of \(D + A\) belonging to \(\mu'\). So \(|X| = \pm X'\). Using Perron–Frobenius’ theorem again, we have

\[X' > 0, \quad |X| = X', \quad \text{and} \quad |x_i| > 0 \ (i = 1, 2, \ldots, n).\]

Since

\[\sum_{v_i \sim v_j, \ i < j} (x_i - x_j)^2 = \sum_{v_i \sim v_j, \ i < j} (|x_i| + |x_j|)^2\]

and

\[(x_i - x_j)^2 \leq (|x_i| + |x_j|)^2,\]

hence

\[(x_i - x_j)^2 = (|x_i| + |x_j|)^2\]

when \(v_i \sim v_j\). Therefore \(x_i x_j < 0\) if \(v_i \sim v_j\).

Let \(V_1 = \{v_i \mid x_i > 0\}, V_2 = \{v_j \mid x_j < 0\}\). For each edge \(e = (v_i, v_j)\), we have \(x_i x_j < 0\). One of the vertices of edge \(e\) is in \(V_1\), the other is in \(V_2\). So \(G\) is a bipartite graph.

(3) From Merris’ result [1], when \(G\) is a bipartite graph, we get \(u = u'\). \(\square\)

**Lemma 3** [6]. If \(G\) is a connected simple graph, then

\[\rho(G) \leq \frac{d_n - 1 + \sqrt{(d_n + 1)^2 + 4(2m - d_n n)}}{2},\]

the equality holds if and only if \(G\) is a regular graph or a bidegreed graph in which each vertex is of degree either \(d_n\) or \(n - 1\).
Let $L_G$ be the line graph of $G$. $L_G$ has $n'$ vertices, $m'$ edges and minimal degree $d'$ in $L_G$, where

$$d' = \min \{d_G(u) + d_G(v) - 2 \mid uv \in E(G)\}.$$

**Lemma 4.** Let $G$ be a connected bipartite graph and $L_G$ has the minimal degree $d' = 2d_n - 2$.
1. If $L_G$ is a regular graph, then $G$ is a regular bipartite graph.
2. $L_G$ is not a bidegreed graph in which each vertex is of degree either $d'$ or $n' - 1$.

**Proof.** 1. Let $(V_1, V_2)$ be a bipartition of graph $G$. Since $G$ is a connected graph, $L_G$ is a connected graph too. From the fact that $L_G$ is a connected regular graph and $G$ is a connected bipartite graph, we can get that $G$ is a semiregular graph. We assume that $d_G(v_i) = r$ when $v_i \in V_1$ and $d_G(u_j) = s$ when $u_j \in V_2$. Without loss of generality, we suppose $r \geq s$. Hence $d_n = s$, $d' = r + s - 2 = 2d_n - 2$ and $r + s = 2d_n$, $r = s = d_n$. So $G$ is a regular bipartite graph.

2. If there exists $G$ such that $L_G$ is a bidegreed graph in which each vertex is of degree either $d'$ or $n' - 1$. Suppose $e = (x, y) \in E(G)$, $e \in V(L_G)$ and $d_{L_G}(e) = n' - 1$. So each edge of $G$ must have a same vertex with edge $xy$. Hence, each vertex of $G$ must be adjacent with $x$ or $y$. Because $G$ is a bipartite graph, vertices $u_j$ ($j = 1, 2, \ldots, r$) adjacent with $x$ cannot be adjacent with $y$, vertices $v_i$ ($i = 1, 2, \ldots, s$) adjacent with $y$ cannot be adjacent with $x$. Hence $G$ is only isomorphic to bistar graph $G(r, s)$ ($G$ cannot be isomorphic to star graph since the line graph of star graph is complete graph). We assume $r \geq s$. Therefore $d_n = 1$, $d' = s \geq 1$, and $d' \neq 2d_n - 2$. It is contradictory and part 2 is proved. □

**Lemma 5** [6]. $f(x) = x - 1 + \sqrt{(x + 1)^2 + 4(2m - xn)}$ is a decreasing function of $x$ for $1 \leq x \leq n - 1$, where $n - 1 \leq m \leq n(n - 1)/2$ and $2m \geq xn$.

**Theorem 1.** If $G$ is a connected simple graph, then

$$\mu(G) \leq d_n + \frac{1}{2} + \sqrt{\left(d_n - \frac{1}{2}\right)^2 + \sum_{i=1}^{n} d_i(d_i - d_n)},$$

the equality holds if and only if $G$ is a regular bipartite graph.
Proof. 1. From the fact that $G$ and $L_G$ are connected graphs and Lemma 3, we have

$$\rho(L_G) \leq \frac{d' - 1 + \sqrt{(d' + 1)^2 + 4(2m' - d'n')}}{2}.$$  

We know

$$n' = m = \frac{1}{2} \sum_{i=1}^{n} d_i, \quad 2m' = \sum_{i=1}^{n} d_i(d_i - 1), \quad d' \geq 2d_n - 2.$$  

By Lemma 5, we get

$$\rho(L_G) \leq d_n - \frac{3}{2} + \sqrt{(d_n - \frac{1}{2})^2 + \sum_{i=1}^{n} d_i(d_i - d_n)}.$$  

Using Lemma 2, we obtain

$$\mu(G) \leq 2 + \rho(L_G) \leq d_n + \frac{1}{2} + \sqrt{(d_n - \frac{1}{2})^2 + \sum_{i=1}^{n} d_i(d_i - d_n)}.$$  

2. When the equality holds, we have

$$\mu(G) = 2 + \rho(L_G) = d_n + \frac{1}{2} + \sqrt{(d_n - \frac{1}{2})^2 + \sum_{i=1}^{n} d_i(d_i - d_n)}$$  

and $d' = 2d_n - 2$.

By Lemma 2 and $\mu(G) = 2 + \rho(L_G)$, we know $G$ is a connected bipartite graph.

By Lemma 3 and

$$2 + \rho(L_G) = d_n + \frac{1}{2} + \sqrt{(d_n - \frac{1}{2})^2 + \sum_{i=1}^{n} d_i(d_i - d_n)},$$

we get $L_G$ is a regular graph or a bidegreed graph in which each vertex is of degree either $d'$ or $n' - 1$.

By Lemma 4, $G$ is only isomorphic to a regular bipartite graph.

3. If $G$ is a regular bipartite graph, $d_i = d_n = r$, $L_G$ is a regular graph and $d' = 2r - 2$. $\rho(L_G) = 2r - 2$, $\mu(G) = 2 + \rho(L_G) = 2r$.

$$d_n + \frac{1}{2} + \sqrt{(d_n - \frac{1}{2})^2 + \sum_{i=1}^{n} d_i(d_i - d_n)} = 2r.$$  

The equality holds. □
Corollary 1. If $G$ is a connected graph, then

$$
\mu(G) \leq \frac{d'}{2} + \frac{3}{2} + \left(\frac{d' + 1}{2}\right)^2 + \sum_{i=1}^{n} d_i \left(\frac{d_i - d'}{2} - 1\right),
$$

the equality holds if and only if $G$ is a semiregular graph or a bistar graph $G(r, r)$.

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References