Algebraic connectivity and doubly stochastic tree matrices

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Abstract

In this paper, we investigate the relations between the smallest entry of a doubly stochastic tree matrix associated with a tree and the diameter of the tree, which are used to deal with Merris's conjecture on the algebraic connectivity and the smallest entries. Further, we present a new upper bound for algebraic connectivity in terms of the smallest entry, which improves Merris' result.

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1. Introduction

Let \( G = (V, E) \) be a simple graph with vertex set \( V(G) = \{v_1, \ldots, v_n\} \) and edge set \( E(G) \). Denote by \( d(v_i) \) or \( d_i \) the degree of vertex \( v_i \), and \( D(G) = \text{diag}(d_1, \ldots, d_n) \) the degree diagonal matrix, \( A(G) \) the \( n \times n \) adjacency matrix whose \((i, j)\)-entry is 1 if \((v_i, v_j) \in E\) and 0 otherwise. The matrix \( L(G) = D(G) - A(G) \) is called the Laplacian matrix of \( G \), which has been extensively studied in the past 20 years (e.g. [11,15,19] and the references therein). It is
obvious that $L(G)$ is singular and positive semidefinite. Thus, its eigenvalues can be arranged as $\lambda_1(G) \geq \lambda_2(G) \geq \cdots \geq \lambda_n(G) = 0$. The second smallest eigenvalue $\lambda_{n-1}(G)$, also denoted $\alpha(G)$, is known as the algebraic connectivity of $G$ (see \cite{6,11}). It is well known that $\alpha(G) > 0$ if and only if $G$ is connected. Since the algebraic connectivity is relevant to the diameter of graphs, the expanding properties of graphs, the combinatorial optimization problems and the theory of elasticity, etc (for example, see \cite{14,15}), it has received much more attention. Recently, there is an excellent survey on algebraic connectivity of graphs written by de Abreu \cite{1}, which is referred to the reader for further information.

In the study of chemical information processing, Goiender et al. \cite{7} introduced another important matrix: doubly stochastic graph matrix associated with a graph, which may be used to describe some properties of topological structure of chemical molecular. Let $I_n$ be the $n \times n$ identity matrix and $\Omega(G) = (I_n + L(G))^{-1} = (\omega_{ij})$. It is easy to see (\cite{7} or \cite{12}) that $\Omega(G)$ is a doubly stochastic matrix. Thus $\Omega(G)$ is called the doubly stochastic graph matrix. On the other hand, Chebotarev and Shamis \cite{4} and Chebotarev \cite{5} pointed out that the doubly stochastic graph matrix may be used to measure the proximity among vertices and evaluate the group cohesion in the construction of sociometric indices and represent a random walk. Merris in \cite{12,13}, Zhang and Wu \cite{17} and Zhang \cite{18} investigated properties of doubly stochastic graph matrices, respectively. Moreover, Pereira \cite{16} studied the spectra of doubly stochastic matrices. The following two conjectures are proposed by Merris \cite{13}.

**Conjecture 1.1.** Let $G$ be a graph on $n$ vertices. Then

$$\alpha(G) \geq 2(n + 1)\omega(G),$$

where $\omega(G)$ is the smallest entry of $\Omega(G) = (\omega_{ij})$, i.e., $\omega(G) = \min\{\omega_{ij}; 1 \leq i, j \leq n\}$.

**Conjecture 1.2.** Let $E_n$ be the degree anti-regular graph, i.e., the unique connected graph whose vertex degrees attain all values between 1 and $n - 1$. Then

$$\omega(E_n) = \frac{1}{2(n + 1)}.$$

In 2000, Berman and Zhang \cite{3} confirmed Conjecture 1.2, while Zhang and Wu \cite{17}, in 2005, presented an example to illustrate that Merris’ Conjecture 1.1 does not hold generally. Motivated by the two conjectures and the related results, we hope further to investigate the relations between the algebraic connectivity and the smallest entry. In Section 2, we investigate the relations between the smallest entries of doubly stochastic matrix of a tree and the diameter of the tree. In Section 3, we present here some examples to show that Conjecture 1.1 does not hold for some trees, and we also present some additional conditions for it to be true. Finally, we obtain a new upper for algebraic connectivity in terms of the smallest entry, which improves Merris’ result.

## 2. The smallest entry and diameter

In this section, we discuss relations between the smallest entry and diameter. First, we give an upper bound for the smallest entries.

**Theorem 2.1.** Let $G$ be a tree with vertex set $V(T) = \{v_1, \ldots, v_n\}$. Assume that there is an edge $e = v_{s+1}v_{s+2} \in E(T)$ such that $F_1$ with $V(F_1) = \{v_1, \ldots, v_{s+1}\}$ and $F_2$ with $V(F_2) = \{v_{s+2}, \ldots, v_n\}$ are two components of $F = T - e$. Denote by $\Omega(F_1) = (\omega_{ij}')$ for $1 \leq i, j \leq s +$
1, $\Omega(F_2) = (\omega'_{ij})$ for $s + 2 \leq i, j \leq n$ and $\Omega(T) = (\omega_{ij})$ for $1 \leq i, j \leq n$. Then the smallest entry of $\Omega(T)$ satisfies

$$\omega(T) \leq \min \left\{ \omega(F_1), \omega(F_2), \frac{(1 - \omega'_{s+1,s+1})(1 - \omega'_{s+2,s+2})}{s(n - s - 2)(1 + \omega'_{s+1,s+1} + \omega'_{s+2,s+2})} \right\}. \quad (1)$$

**Proof.** Let $x_i$ be a vector of $n$ dimension whose only nonzero component is 1 in the $i$th and $x = x_{s+1} - x_{s+2}$. Thus $L(T) = L(F) + xx^T$. By the Sherman–Morrison formal (see, e.g. [9, p. 19]), we have

$$\Omega(T) = \Omega(F) - \frac{\Omega(F)xx^T \Omega(F)^T}{1 + x^T \Omega(F)x}. \quad (2)$$

Clearly, $x^T \Omega(F)x = \omega'_{r+1,r+1} + \omega'_{r+2,r+2}$ and

$$\Omega(F)xx^T \Omega(F) = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix},$$

where $A_{11} = (\omega_{1,s+1}, \ldots, \omega_{s+1,s+1})^T(\omega_{1,s+1}, \ldots, \omega_{s+1,s+1}) > 0$ and $A_{12} = -(\omega_{1,s+1}, \ldots, \omega_{s+1,s+1})^T(\omega_{s+2,s+2}, \ldots, \omega_{s+2,n}) < 0$. Hence

$$\omega(T) < \omega(F_1), \quad \omega(T) < \omega(F_2). \quad (3)$$

Moreover

$$\omega(T) \leq \frac{\omega'_{i,s+1} \omega'_{j,s+2}}{1 + \omega'_{s+1,s+1} + \omega'_{s+2,s+2}}, \quad 1 \leq i \leq s + 1, \quad s + 2 \leq j \leq n. \quad (4)$$

Let $\omega'_{k,s+1} = \min[\omega'_{i,s+1}, 1 \leq i \leq s + 1]$ and $\omega'_{j,s+2} = \min[\omega'_{j,s+2}, s + 2 \leq j \leq n]$. Then

$$1 = \sum_{i=1}^{s+1} \omega'_{i,s+1} \geq s\omega'_{k,s+1} + \omega'_{s+1,s+1},$$

which implies that $\omega'_{k,s+1} \leq \frac{1 - \omega'_{s+1,s+1}}{s}$. Similarly, $\omega'_{j,s+2} \leq \frac{1 - \omega'_{s+2,s+2}}{n-s-2}$. Hence by (4) and Lemma 2.2 in [18], we have

$$\omega(T) \leq \frac{\frac{1 - \omega'_{s+1,s+1}}{s} \times \frac{1 - \omega'_{s+2,s+2}}{n-s-2}}{1 + \frac{1 - \omega'_{s+1,s+1}}{s} + \frac{1 - \omega'_{s+2,s+2}}{n-s-2}} = \frac{(1 - \omega'_{s+1,s+1})(1 - \omega'_{s+2,s+2})}{s(n - s - 2)(1 + \omega'_{s+1,s+1} + \omega'_{s+2,s+2})}.$$}

So the assertion holds. $\square$

Denote by $T_{s,t}$ the tree of order $n$ obtained by joining two centers of the two star graphs $K_{1,s}$ and $K_{1,t}$ with $s + t + 2 = n$ and $s \geq 1, t \geq 1$.

**Corollary 2.2.** Let $T$ be a tree of order $n \geq 4$ rather than the star graph $K_{1,n-1}$. Then

$$\omega(T) \leq \frac{1}{5n + 1}$$

with equality if and only if $T$ is $T_{n-3,1}$.

**Proof.** Since $T$ is not $K_{1,n-1}$, there exists an edge $e = v_{s+1}v_{s+2}$ such that $F_1$ with $V(F_1) = \{v_1, \ldots, v_{s+1}\}$ and $F_2$ with $V(F_2) = \{v_{s+2}, \ldots, v_n\}$ are two components of $F = T - e$ and $1 \leq s \leq n - 3$. By Theorem 1 in [13], we have
\[
\omega'_{s+1,s+1} \geq \frac{2}{s+2}, \quad \omega'_{s+2,s+2} \geq \frac{2}{n-s}
\]
with each equality if and only if \(F_1\) is the star graph \(K_{1,s}\) with center \(v_{s+1}\) and \(F_2\) is the star graph \(K_{1,n-s-2}\) with center \(v_{s+2}\), respectively. By Theorem 2.1

\[
\omega(T) \leq \frac{(1 - \omega'_{s+1,s+1})(1 - \omega'_{s+2,s+2})}{s(n-s-2)(1 + \omega'_{s+1,s+1} + \omega'_{s+2,s+2})}
\]
\[
\leq \frac{(1 - \frac{2}{s+2})(1 - \frac{2}{n-s})}{s(n-s-2) \left(1 + \frac{2}{s+2} + \frac{2}{n-s}\right)}
\]
\[
\leq \frac{1}{(s+2)(n-s) + 2n + 4}
\]
\[
\leq \frac{1}{5n+1}.
\]

If equality holds, then \(F_1\) and \(F_2\) are two star graphs \(K_{1,s}\) and \(K_{1,n-s-2}\), respectively. Moreover, \(s = 1\) or \(s = n-3\). Therefore, \(T\) must be \(T_{n-3,1}\). If \(T\) is \(T_{n-3,1}\), by a simple calculation, it is easy to see that

\[
\omega(T_{n-3,1}) = \frac{1}{5n+1}.
\]

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**Corollary 2.3.** Let \(T\) be of a tree \(T_{s,n-s-2}\) of order \(n\) with \(1 \leq s \leq n-3\). Then

\[
\omega(T_{s,n-s-2}) = \frac{1}{(s+4)(n-s+2) - 4}
\]

and

\[
\omega(T_{1,n-3}) > \omega(T_{2,n-4}) > \cdots > \omega(T_{\lfloor \frac{n}{2} \rfloor - 1, \lceil \frac{n}{2} \rceil - 1}).
\]

**Proof.** Let \(e\) be an edge such that \(T - e\) has two components with \(K_{1,s}\) and \(K_{1,n-s-2}\). By formula (2) and some calculation, we obtain

\[
\omega(T_{s,n-s-2}) = \frac{1}{(s+4)(n-s+2) - 4}.
\]

Moreover, \(\omega(T_{s,n-s-2})\) decrease strictly with respect to \(1 \leq s \leq \lfloor \frac{n}{2} \rfloor - 1\).

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**Corollary 2.4.** Let \(T\) be a tree of order \(n\) with diameter \(d\). Then

\[
\omega(T) \leq \frac{\sqrt{5}}{\left(\frac{3+\sqrt{5}}{2}\right)^{d+1} - \left(\frac{3-\sqrt{5}}{2}\right)^{d+1}}
\]

with equality if and only if \(T\) is a path of order \(n\).

**Proof.** We notice that \(T\) can be built from a path \(P_{d+1}\) of order \(d+1\) by attaching pendant vertices. It follows from Theorem 2.1 that this building process strictly decrease the smallest entry \(\omega(T) \leq \omega(P_{d+1})\) with equality if and only if \(T\) is a path. On the other hand, by Theorem 2.1 in [17], we have

\[
\omega(P_{d+1}) = \frac{\sqrt{5}}{\left(\frac{3+\sqrt{5}}{2}\right)^{d+1} - \left(\frac{3-\sqrt{5}}{2}\right)^{d+1}}.
\]

So the assertion holds. \(\Box\)
Lemma 2.5. Let $T$ be a tree of order $n$ with vertex set $V(T) = \{v_1, \ldots, v_n\}$. If $\omega(T) = \min\{\omega_{ij}, 1 \leq i, j \leq n\} = \omega_{k,l}$, then

(1) $v_k$ and $v_l$ are two pendant vertices, i.e., the degrees of vertices $v_k$ and $v_l$ are 1.
(2) If diameter $d$ of $T$ is no more than 4, then the distance between $v_k$ and $v_l$ is equal to $d$.

Proof. (1) If $v_k$ is not a pendant vertex, there exists a pendant vertex $v_i$ such that there is a path $P_{il}$ from $v_i$ to $v_l$ containing $v_k$. By Theorem 3.2 in [10] and Theorem 2 in [13], we have

$$\omega_{il} = \frac{\omega_{ik}\omega_{kl}}{\omega_{kk}} \leq \frac{\omega_{ik}\omega_{kl}}{2\omega_{ki}} < \omega_{kl},$$

a contradiction, consequently, $v_k$ is a pendant vertex. Similar arguments applied to $v_l$. We can show that $v_l$ is also a pendant vertex.

(2) If $d = 2$, it follows from (1) the assertion holds. If $d \geq 3$ and the distance between $v_k$ and $v_l$ is 2, then there exist two paths $v_kv_i v_j \cdots v_r$ of length $d$ and $v_kv_l v_j$ of length 2, where $v_r$ is a pendant vertex. By Theorem 3.2 in [10] and Theorem 2 in [13], we have

$$\omega_{kr} = \frac{\omega_{ki}\omega_{il}}{\omega_{ii}} < \frac{\omega_{ki}\omega_{lr}}{2\omega_{ir}} = \frac{\omega_{ki}}{2}$$

and

$$\omega_{kl} = \frac{\omega_{kj}\omega_{jl}}{\omega_{jj}} = \frac{\omega_{kj}}{2}.$$ 

Thus $\omega_{kr} < \omega_{kl}$. It is impossible and the distance between $v_k$ and $v_l$ is at least 3. Therefore, if $d = 3$, then the distance between $v_k$ and $v_l$ is equal to $d$.

If $d = 4$, then the distance $d(v_k, v_l)$ between vertex $v_k$ and $v_l$ is at least 3. Suppose that $d(v_k, v_l) = 3$. Since $v_k$ and $v_l$ are two pendant vertices and $d = 4$, without loss of generality, we assume that there exist two paths $v_kv_i v_j v_p v_q$ and $v_k v_l v_i v_j v_l$. By Theorem 3.2 in [10] and Theorem 2 in [13], we have

$$\omega_{kl} = \frac{\omega_{kj}\omega_{jl}}{\omega_{jj}} = \frac{\omega_{kj}}{2}.$$ 

and

$$\omega_{kl} = \frac{\omega_{kj}\omega_{jq}}{\omega_{jj}} < \frac{\omega_{kj}}{2}.$$ 

Hence $\omega_{kl} < \omega_{kl}$, a contradiction. Therefore, $d(v_k, v_l) \geq 4$ and the assertion holds. \[\Box\]

Example 2.6. Let $T$ be a tree of order $n = 2s + 6$ as in Fig. 1.

By Lemma 2.5, $\omega(T) = \min\{\omega_{16}, \omega_{18}\}$. By Theorem 2 in [13] and Lemma 2.2 in [20], we have

$$\omega_{37} \leq \frac{\omega_{77}}{2} \leq \frac{1}{2} \times \frac{2}{s+3} = \frac{1}{s+3}.$$ 

On the other hand, by Lemmas 3.1 and 2.2 in [20], we have $\omega_{34} > \frac{1}{11}, \omega_{45} > \frac{1}{8}$ and $\omega_{44} \leq \frac{2}{4}$.

Hence by Theorem 3.2 in [10] and Theorem 2 in [13], we have

$$\omega_{36} = \frac{\omega_{34}\omega_{45}\omega_{56}}{\omega_{44}\omega_{55}} > \frac{1}{11} \times \frac{1}{8} \times \omega_{56} = \frac{1}{88}.$$
If \( s \geq 41 \), then by Theorem 3.2 in [10] and Theorem 2 in [13],
\[
\omega_{18} = \frac{\omega_{13} \omega_{37} \omega_{78}}{\omega_{33} \omega_{77}} = \frac{\omega_{13} \omega_{37}}{2 \omega_{33}} \leq \frac{\omega_{13} \times \frac{1}{s+1}}{2 \omega_{33}} \leq \frac{\omega_{13}}{\omega_{33}} \leq \frac{\omega_{13} \omega_{36}}{\omega_{33}} = \omega_{16}.
\]
Therefore, if \( s \geq 41 \), we have \( \omega(T) = \omega_{18} \) and the distance \( d(v_1, v_8) = 4 \), while the diameter \( d = 5 \).

3. Algebraic connectivity and the smallest entry

In this section, we first show that Conjecture 1.1 does not hold for \( T_{s,n-s-2} \) with \( n \geq 12 \).

**Theorem 3.1.** Let \( T \) be a tree \( T_{s,n-s-2} \) of order \( n \) with diameter 3. If \( T_{s,n-s-2} \) is one of \( T_{2,2}, T_{1,8}, T_{1,7}, \ldots, T_{1,1} \), then
\[
\alpha(T_{s,n-s-2}) \geq 2(n+1)\omega(T_{s,n-s-2}).
\]
If \( T_{s,n-s-2} \) is not any one of \( T_{2,2}, T_{1,8}, T_{1,7}, \ldots, T_{1,1} \), then
\[
\alpha(T_{s,n-s-2}) < 2(n+1)\omega(T_{s,n-s-2}).
\]

**Proof.** Without loss of generality, we assume that \( 1 \leq s \leq \lfloor \frac{n}{2} \rfloor - 1 \). By Proposition 1 in [8], \( \alpha(T_{s,n-s-2}) \) is the only root of equation \( f(x) = x^3 - (n + 2)x^2 + (2n + s(n - s - 2) + 1)x - n = 0 \) in the interval \([0, 1]\). By Corollary 2.3, we have
\[
\omega(T_{s,n-s-2}) = \frac{1}{(s+4)(n-s+2)-4} = \frac{s(n-s-2)+4n+4}{s(n-s-2)+4n+4}.
\]
Clearly, \( f(0) = -n < 0 \). Now we consider the following four cases:

If \( s \geq 2 \) and \( n \geq 7 \), then \( s(n - s - 2) \geq 2(n - 4) \) and
\[
\begin{align*}
&f(2(n+1)\omega(T_{s,n-s-2})) \\
&> -(n+2) \left( \frac{2(n+1)}{s(n-s-2)+4n+4} \right)^2 \\
&\quad + (2n + s(n - s - 2) + 1) \frac{2(n+1)}{s(n-s-2)+4n+4} - n \\
&\quad = \frac{(s(n-s-2)+4n+4)((n+2)s(n-s-2)+2n+2) - 4(n+2)(n+1)^2}{(s(n-s-2)+4n+4)^2} \\
&\quad \geq \frac{(6n-4)(2n^2-2n-14) - 4(n+2)(n+1)^2}{(s(n-s-2)+4n+4)^2} \\
&\quad = \frac{4(2n^3-9n^2-24n+12)}{(s(n-s-2)+4n+4)^2} > 0.
\end{align*}
\]
If \( s = 2 \) and \( n = 6 \), then \( f(2(n + 1)\omega(T_{2,n-4})) < 0 \).
If \( s = 1 \) and \( n \geq 12 \), then \( 2(n + 1)\omega(T_{s,n-s-2}) = \frac{2(n+1)}{5n+1} \) and
\[
\begin{align*}
f(2(n + 1)\omega(T_{1,n-3})) &= \left( \frac{2(n + 1)}{5n + 1} \right)^3 - (n + 2) \left( \frac{2(n + 1)}{5n + 1} \right)^2 \\
&\quad + (3n - 2) \frac{2(n + 1)}{5n + 1} - n \\
&= \frac{5n^4 - 43n^3 - 181n^2 - 75n - 4}{(5n + 1)^3} > 0.
\end{align*}
\]

If \( s = 1 \) and \( 4 \leq n \leq 11 \), then \( f(2(n + 1)\omega(T_{1,n-3})) < 0 \).
Hence if \( T \) is one of \( T_{2,2}, T_{1,8}, T_{1,7}, \ldots, T_{1,1} \), then \( f(2(n + 1)\omega(T)) < 0 \), which implies that \( \alpha(T) \geq 2(n + 1)\omega(T) \).
If \( T \) is not any one of \( T_{2,2}, T_{1,8}, T_{1,7}, \ldots, T_{1,1} \), then \( f(2(n + 1)\omega(T)) > 0 \), which implies that \( \alpha(T) < 2(n + 1)\omega(T) \).
So the assertion holds.

**Theorem 3.2.** Let \( T \) be a tree of order \( n \geq 4 \) with diameter \( d \). If \( d \geq \frac{\lg 3 + 3\lg n}{\lg(3+\sqrt{5})-\lg 2} - 1 \), then \( \alpha(T) \geq 2(n + 1)\omega(T) \).

**Proof.** Since \( d \geq \frac{\lg 3 + 3\lg n}{\lg(3+\sqrt{5})-\lg 2} - 1 \), we have
\[
\left( \frac{3 + \sqrt{5}}{2} \right)^{d+1} > 3n^3 \geq \sqrt{5}n^2(n + 1).
\]
Hence by Theorem 4.2 in [14] and Corollary 2.4
\[
\alpha(T) \geq \frac{4}{nd} > 4(n + 1)\sqrt{5} \left( \frac{3 + \sqrt{5}}{2} \right)^{-d+1}
\geq \frac{2(n + 1)\sqrt{5}}{\left( \frac{3+\sqrt{5}}{2} \right)^{d+1} - \left( \frac{3-\sqrt{5}}{2} \right)^{d+1}}
\geq 2(n + 1)\omega(T).
\]
So the assertion holds.

**Remark.** From the above results, we may see that Conjecture 1.1 holds for many trees, since the diameter of any random trees is almost equal to \( O(\lg n) \). While it does not holds for smaller diameter and larger order. The following Lemma will be used to give a new upper bound.

**Lemma 3.3.** Let \( T \) be a tree of order \( n \geq 3 \). If \( v_r \) is not a pendant vertex, then \( \omega_{rj} \geq 2\omega(T) \) for \( j = 1, \ldots, n \) and with equality for all \( 1 \leq j \neq r \leq n \) if and only if \( T \) is the star graph \( K_{1,n-1} \).

**Proof.** Let \( \omega(T) = \min\{\omega_{ij}, 1 \leq i, j \leq n\} = \omega_{kl} \). By Lemma 2.5, both \( v_k \) and \( v_l \) are pendant vertices with \( v_r \neq v_k, v_l \). Let \( P(v_k, v_l) \) be the only one path from \( v_k \) to \( v_l \). By Theorem 2 in [13], we consider the following two cases:

**Case 1:** \( P(v_k, v_l) \) contains vertex \( v_r \). If \( v_j \) is contained in the path \( P(v_k, v_r) \), then by Theorem 3.2 in [10] and Theorem 2 in [13], we have
\[ \omega_{rj} = \frac{\omega_{rk}\omega_{jj}}{\omega_{jr}} \geq 2\omega_{rk} = \frac{2\omega_{rr}\omega_{jk}}{\omega_{jr}} \geq 4\omega(T) > 2\omega(T). \]

If \( v_j \) is contained in the path \( P(v_l, v_r) \), then by Theorem 3.2 in [10] and Theorem 2 in [13], we have
\[ \omega_{rj} = \frac{\omega_{rl}\omega_{jj}}{\omega_{jl}} \geq 2\omega_{rl} = \frac{2\omega_{rr}\omega_{kl}}{\omega_{kr}} \geq 4\omega(T) > 2\omega(T). \]

If \( v_j \) is not contained in the path \( P(v_k, v_l) \), then either \( P(v_j, v_k) \) or \( P(v_j, v_l) \) contains vertex \( v_r \), say, \( v_r \) is contained in \( P(v_j, v_k) \). Hence by Theorem 3.2 in [10] and Theorem 2 in [13]
\[ \omega_{rj} = \frac{\omega_{kj}\omega_{rr}}{\omega_{kr}} \geq 2\omega_{kj} \geq 2\omega(T). \] (5)

**Case 2:** \( P(v_k, v_l) \) does not contain vertex \( v_r \). Clearly there exists an edge \( e = v_r v_s \) such that \( F_1 \) and \( F_2 \) are two components of \( T - e \) and \( F_1 \) contains at least one edge \( v_r v_l \). For any vertex \( v_j \in V(F_1) \), by Theorem 3.2 in [10] and Theorem 2 in [13], we have
\[ \omega_{rj} = \frac{\omega_{kj}\omega_{rr}}{\omega_{kr}} > 2\omega_{kj} \geq 2\omega(T), \]

since \( v_j \) is not adjacent to vertex \( v_k \). Similarly, for any vertex \( v_j \in V(F_2) \), by Theorem 3.2 in [10] and Theorem 2 in [13], we have
\[ \omega_{rj} = \frac{\omega_{jl}\omega_{rr}}{\omega_{rl}} > 2\omega_{jl} \geq 2\omega(T). \]

Therefore, \( \omega_{rj} \geq 2\omega(T) \) for \( j = 1, \ldots, n \).

If \( T \) is the star graph \( K_{1,n-1} \), the equation in the formula holds for all \( j \neq r \). Conversely, assume that \( \omega_{rj} = 2\omega(T) \) for all \( 1 \leq j \neq r \leq n \). Then from the above proof, it is easy to see that \( P(v_k, v_l) \) contains vertex \( v_r \) and equalities in (5) hold. Then \( \omega_{rk} = \omega_{rl} = 2\omega(T) \). Hence
\[ \omega_{kl} = \frac{\omega_{kr}\omega_{rl}}{\omega_{rr}} \geq \frac{\omega_{kr}}{2} = \omega_{kl}. \]

Therefore, \( v_r \) is adjacent to \( v_k \). Similarly, \( v_r \) is adjacent to \( v_l \). Hence the distance between \( v_k \) and \( v_l \) is equal to 2. By Lemma 2.5, the diameter of \( T \) is equal to 2 which implies that \( T \) is the star graph \( K_{1,n-1} \). \( \square \)

**Theorem 3.4.** Let \( T \) be a tree of order \( n \) with \( p \) non-pendant vertices. Then
\[ \alpha(T) \geq \frac{(n + p)\omega(T)}{1 - (n + p)\omega(T)} \]
with equality if and only if \( T \) is the star graph \( K_{1,n-1} \).

**Proof.** Clearly, \( \frac{1}{1+\alpha(T)} \) is the second largest eigenvalue of \( \Omega(T) \). By Theorem 2.5.10 in [2],
\[ \frac{1}{1+\alpha(T)} \leq 1 - \sum_{i=1}^{n} c_i, \]
where \( c_i \) is the smallest entry of the \( i \)th column of \( \Omega(T) \). By Lemma 3.3, \( c_i \geq 2\omega(T) \) for \( v_i \) is not pendant vertex. Hence \( \frac{1}{1+\alpha(T)} \leq 1 - (n + p)\omega(T) \). So the assertion holds.

If equality holds, then \( c_i = 2\omega(T) \) for \( v_i \) is not pendant vertex. By Lemma 3.3, \( T \) is the star graph \( K_{1,n-1} \). Conversely, by a simple calculation, it is easy to see that \( \omega(K_{1,n-1}) = \frac{1}{2(n+1)} \) and \( \alpha(K_{1,n-1}) = 1 \). So \( \frac{(n+p)\omega(T)}{1-(n+p)\omega(T)} = 1 = \alpha(K_{1,n-1}) \). \( \square \)
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References