Influence of the nearest-neighbor connections on shaping weighted evolving network

Yikang Rui\(^a\), Wenzhou Wu\(^b\), Dingtao Shen\(^a,c\), Jiechen Wang\(^a\)

\(^a\) Jiangsu Provincial Key Laboratory of Geographic Information Science and Technology, Nanjing University, 210093 Nanjing, China
\(^b\) State Key Laboratory of Resources and Environmental Information System, Chinese Academy of Sciences, 100101 Beijing, China
\(^c\) Changjiang River Scientific Research Institute, Changjiang Water Resources Commission, 430010 Wuhan, China

A R T I C L E   I N F O

Article history:
Received 26 March 2014
Accepted 18 September 2014
Available online 27 October 2014

A B S T R A C T

This paper proposes an extended local-world evolving network model consisting of global strength-driven preferential attachment for one central node, and local weight-driven preferential attachment for nearest neighbors of the central node. Analytical predictions and numerical simulations were executed for network evolutions and distributions. The obtained power-law behaviors display the same exponent functions as the ones in a classic model. More comparisons between these two models were made to investigate the structural differences that the nearest-neighbor connections result in. Compared with the counterpart, the proposed model shows a higher clustering coefficient, the varying average shortest path length and the significant hierarchical organization. Our model is generally robustness and yet fragility, and is weaker in synchronizability than the counterpart. All those results are added to our understanding of how the rule of the nearest-neighbor connections affects the characteristics of weighted evolving network.

1. Introduction

The studies of social, biological, economic and numerous other networks have been added to our understandings of complex networks, which exhibit significant small-world property and scale-free behavior [1–6]. Many real-world networks are represented as weighted networks [7–11], thus, a amount of evolving weighted network models have been built to investigate the non-trivial correlation between edge weight and topological quantity [12–14]. One classic growing model for weighted networks was proposed by Barrat, Barthélemy and Vespignani (BBV) [15,16]. The BBV model is based on the mechanisms of strength preferential attachment and weight dynamical evolution. It displays scale-free behaviors for the distributions of node degree, strength and edge weight. Many extended models were later designed by adding new evolution rules including traffic-driven growth [17], spatial constraints [18], group-based preferential attachment [19] and accelerating growth [20–22].

In many real-world networks such as the world trade web [23] and the Internet on router level [24], it is difficult for one node to obtain global information about the entire network. A local-world evolving network model was therefore introduced to describe the preferential attachment mechanism on the local level [25]. To improve the low clustering coefficient, a triad formation step was later added into local preferential attachment [26,27]. The local world usually consists of randomly chosen nodes. However, taking a social network as the example, we find that one person more easily knows persons in the same community [28]. Suppose edge weight stands for the closeness between two persons, and a new individual \(A\) has established a relationship with the individual \(B\) who is famous (large value of strength). Then \(A\) has a high probability to know the individual \(C\) who is a close friend of \(B\) based on closeness between the individuals \(C\) and \(B\). We thus build...
the model with one central node and its nearest neighbors as the local world, which is abbreviated as the NNLW model. The central node is selected by global strength-driven preferential attachment while neighbors of the central node to be connected are selected by local weight-driven preferential attachment.

The rest of the paper is organized as follows. Section 2 describes the evolutionary rules of the NNLW model. Section 3 provides the theoretical predictions and corresponding simulation results for the distributions and evolution of node degree, strength and edge weight. Section 4 makes full comparisons between the NNLW model and the BBV model by investigating the clustering coefficient, epidemic spreading and synchronization. Finally, Section 5 summarizes this work.

2. A local-world model with the nearest-neighbor preferential attachment

There are initially $N_0$ completely connected nodes and $e_0$ edges. A new node $n$ with $m$ edges is added at each time step, i.e., $m$ existing nodes will be selected to connect the node $n$.

(i) Global strength-driven preferential attachment. One node $i$ is firstly chosen from the existing network as the central node based on the probability of node strength:

$$\prod_{n-i} = \frac{s_i}{\sum_{j: (i,j)} s_j}. \quad (1)$$

(ii) Local weight-driven preferential attachment. $m-1$ nodes are selected from the nearest neighbors of the central node $i$ ($\Omega_i$) to connect to the new node $n$ according to the probability of edge weight:

$$\prod_{n-i} = \frac{w_{ij}}{\sum_{j: (i,j)} w_{ij}} \quad (2)$$

where $j$ is one of the nearest neighbors of the central node $i$.

The initial weight of each new edge $(n, i)$ is $w_0$. Applying the same rule in the BBV model, our model rearranges the weights on all other edges departing from the node $i$ based on weight ratio:

$$w_{ij} \to w_{ij} + \frac{\delta w_{ij}}{s_i}, \quad j \in \Omega_i. \quad (3)$$

The process leads to $s_i \to s_i + w_0 + \delta$, where $\delta$ is the weight increment of the new edge $(n, i)$. We set $w_0 = 1$ and $\delta$ is constant in our experiment. After weight updating, the topological growth and weight dynamic run for another new node until the desired network size is reached.

3. Analytical calculations and numerical simulations

3.1. Degree and strength distributions

It is obvious that time equals the number of nodes added into the network, i.e., the network totally has $N = t + N_0$ nodes and $mt + e_0$ edges at time $t$. According to topological growth rules, when a new coming node $n$ is added, one existing node $i$ can be chosen either as the central node with probability given by Eq. (1), or as one of the nearest neighbors of the central node $j$ by Eq. (2). The time evolution equation for $k_i$ is:

$$\frac{dk_i}{dt} = \frac{s_i}{\sum_{j: (i,j)} s_j} + (m-1) \sum_{j: (i,j)} \frac{s_j}{\sum_{k: (i,k)} s_k} \frac{w_{ij}}{s_j} = m \frac{s_i}{\sum_{i: (i,j)} s_j}. \quad (4)$$

According to the weight dynamic mechanism, the strength $s_i$ of node $i$ increases when the new node $n$ connects either to $i$ with the increment value of $1 + \delta$, or to the nearest neighbors of node $i$ with the increment value of $\langle w_j/s_i \rangle \delta$. Therefore,

$$\frac{ds_i}{dt} = m \frac{s_i}{\sum_{l: (i,j)} s_l} (1+\delta) + \sum_{j: (i,j)} \left( \frac{s_j}{\sum_{k: (i,k)} s_k} + (m-1) \sum_{k: (i,k)} \frac{s_k}{\sum_{l: (i,l)} s_l} \right) \langle w_j/s_i \rangle \delta$$

$$= m \frac{s_i}{\sum_{l: (i,j)} s_l} (1+\delta) + m \frac{s_i}{\sum_{l: (i,j)} s_l} \delta = m \frac{s_i}{\sum_{l: (i,j)} s_l} (1+2\delta). \quad (5)$$

The total strength increased by each added edge is $2(1+\delta)$, implying $\sum_{l: (i,j)} s_l(t) \approx 2m(1+\delta)t$. Eq. (5) can be solved with the initial condition $s_i(t=0) = m$, yielding $s_i(t) = m(t/i)^{1-2\delta}(i/2^\delta)$. According to Eqs. (4) and (5), $k_i(t) = s_i(t)/(1+2\delta)$ is obtained and displays a proportionality relation $s \sim k$.

Numerical simulations were performed to validate the obtained analytical predictions. Numerical results of time evolution for node strength in the top panel of Fig. 1 are consistent with the theoretical ones. The bottom panel of Fig. 1 shows that the relationship between the strength $s_i$ and degree $k_i$ always a linear, which is consistent with the predicted coefficient $1 + 2\delta$.

Suppose the time is uniformly distributed in $[0, t]$ when the node $i$ is added into the network, and then strength probability distribution is written as:

$$P(s, t) = \frac{1}{t + N_0} \int_0^t \delta[s - s_i(t)] \, dt. \quad (6)$$

where $\delta(x)$ is the Dirac delta function.

Because the infinite size limit $t \to \infty$, node strength distribution is $P(s) \sim s^{-\gamma_s}$ with $\gamma_s = 1 + 2\delta \in (2,3]$. Node degree is linearly related to node strength, thus, the degree distribution $P(k)$ also has a power law form $P(k) \sim k^{-\gamma_k}$ with the exponent $\gamma_k$ identical to $\gamma_s$. Fig. 2 shows the strength probability distributions $P(s)$ for different values of $\delta$. The power-law behaviors of $P(s)$ are consistent with the theoretical predictions.

3.2. Weight distributions

In the process of network growth, the increase of edge weight $w_{ij}$ is induced by a new connection between node $n$ and node $i$ (node $j$) with the following changing rate:

$$\frac{dw_{ij}}{dt} = m \frac{s_i}{\sum_{j: (i,j)} s_j} \langle \delta w_{ij}/s_i \rangle + m \frac{s_j}{\sum_{k: (i,k)} s_k} \langle \delta w_{ij}/s_j \rangle = \frac{\delta}{1+\delta} \frac{w_{ij}}{1+\delta}. \quad (7)$$

Suppose $t_{ij}$ is the time when the edge $(i, j)$ is established. The equation can be solved as $w_{ij}(t) = (t/t_{ij})^{\gamma_w/(1+\delta)}$ with
the initial condition $w_{ij}(t_0) = 1$. Similar to analytical calculation for node strength distribution, the edge weight distribution displays a power law form $P(w) \sim w^{-\gamma_w}$ with the exponent $\gamma_w = (1 + 2\delta)/\delta$. Numerical simulations for edge weight again confirm our analytical predictions in Fig. (3).
4. Comparisons between the NNLW model and the BBV model

From the analysis above, it is clear that the NNLW model has the same distribution and time evolution of node degree, node strength and edge weight as the ones in the BBV model, respectively, even though their growth rules are not the same. In this section, more comparisons will be provided to explore the difference in structure between them.

4.1. Average shortest path length and clustering coefficient

The average shortest path length and clustering coefficient are two quantities to characterize the small-world property of the network \[1,5\]. To describe the interconnectedness, the average shortest path length $L$ is defined as the average length of the shortest paths between any pair of two nodes in the network.

$$L = \frac{1}{N(N - 1)} \sum_{i,j} d_{ij},$$

where $d_{ij}$ denotes the length of shortest path between node $i$ and node $j$, and $N$ is total node number. Average clustering coefficient $C$ describes the abundance of neighboring connection. The clustering coefficient of one given node $i$ is defined as the ratio between existing and potential numbers of neighboring connections of node $i$. The average clustering coefficient of the whole network is therefore given by:

$$C = \frac{1}{N} \frac{1}{\sum_{i} k_i(k_i - 1)/2}. \quad (9)$$

Fig. 4 shows that $L$ decreases while $C$ increases with increasing $\delta$ for both models. Compared with the BBV model, the NNLW model displays a larger $L$ when $\delta$ is small, but $L$ decreases much faster and becomes smaller when $\delta$ is large enough. When $\delta$ is small (e.g., $\delta = 0.1$), the values of node degree or strength do not vary so much compared with the ones when $\delta = 20$. So nodes far away from the hubs (i.e., nodes with high values of degree or strength) have a high possibility to be chosen by a new node; meanwhile not all of the neighbors of the chosen nodes are hubs. Therefore, the rule of the nearest-neighbor connections in the NNLW model constrains the possibility that a new node connects $m$ hubs in this case, and leads to a higher $L$. On the contrary, when $\delta$ becomes large (e.g., $\delta = 20$), values of the node degree or strength tend to

**Fig. 3.** Time evolution of edge weight $w_{ij}(t)$ (top panel) and edge weight distribution (bottom panel) with different values of $\delta$ ($N = 10^4$, $m = 3$). The dashed lines are the theoretical predictions. Results are averaged over 50 independent runs.
concentrate on a few hubs, which are mostly neighbors to each other. Once one of these hubs is chosen by a new node, other hubs as its neighbors are more likely to be chosen. Therefore, the rule of the nearest-neighbor connections in the NNLW model improves the possibility that a new node connects more hubs in this case, and makes the $L$ smaller.

$C$ in the NNLW model is always larger than the one in the BBV model, especially when $\delta$ is small. It is logical since the NNLW model applies the nearest-neighbor connections in local attachment. Hubs have much higher values of node degree or strength when $\delta$ increases to 20, and thus are more likely to be selected during the evolution in the BBV model based on the strength-driven preferential attachment. Because these hubs are mostly neighbors to each other, the $C$ of the BBV model in the case of $\delta = 20$ improves a lot.

The clustering-degree correlation $C(k)$ indicates the average clustering coefficient of nodes who have the same degree $k$. Fig. 5 shows that the nontrivial correlation between clustering coefficient and degree in the NNLW model, i.e., $C(k) \sim k^{-1}$, which suggests a hierarchical structure and is independent of the parameter $\delta$. This $k^{-1}$ behavior has been reported in many real-world systems [12,29,30].

### 4.2. Synchronization robustness and fragility

Synchronization is observed in many real-world systems and has various applications. This section compares the synchronization robustness and fragility between the NNLW model and the BBV model. Both networks were firstly transferred into coupling matrices. $a_{ij} = -k_i$. If there is a connection between node $i$ and $j$ ($i \neq j$), $a_{ij} = a_{ji} = 1$; if not: $a_{ij} = a_{ji} = 0$. Synchronizability of a network was then characterized by the second-largest eigenvalue $\lambda_2$ of the corresponding coupling matrix [31]. Robustness and fragility of synchronization were calculated based on random and intentional removal of a small fraction $f$ ($0 < f \ll 1$).
of nodes from the network, respectively. Intentional removal is executed by removing nodes in a decreasing order of degree.

It has been reported that the scale-free network can remain synchronizability almost unchanged during time evolution. It tolerates random errors due to the significant heterogeneity structure; however it is particularly vulnerable to intentional attacks [31,32]. In a local-world evolving network model, the local preferential attachment mechanism affects the node degree distribution and network heterogeneity. Therefore, the synchronization shows a transition between that of the scale-free network and exponential network [25].

Since both the BBV model and NNLW model are scale-free networks, they are generally ‘robustness and yet fragility’. Increasing the number of connections $m$ can enhance the synchronizability of scale-free networks [31,33]. Fig. 6 shows that the synchronizability of both models is improved by increasing the value of $\delta$. The second-largest eigenvalues in the NNLW model decrease quickly from $-0.2$ to $-1.17$ with increasing $\delta$, but are larger than the counterparts in the BBV model, especially when $\delta = 0.5$. It suggests that synchronizability of the NNLW model network is weaker because of the rule of the nearest-neighbor connections.

Panel (a) of Fig. 6 shows the performance of the robustness. Error tolerance of the NNLW model to random removal is a little more stable than the counterpart since the second-largest eigenvalues of the NNLW model remain almost unchanged when $2\%$ of the nodes are randomly removed. Panel (b) of Fig. 6 indicates that both models are vulnerable to intentional attack. When $\delta$ is large, hubs with extremely high values of node degree emerge. Once those hubs are intentionally removed, the whole network is easily broken into several isolate groups and the synchronization is even destroyed for both models. Although those two models have the same degree distribution, the different performance in synchronizability suggests that clustering coefficient is an important indicator to network synchronization.

5. Conclusions

This paper presents an extended local-world evolving network model which combining both global and local preferential attachment. The local world in the NNLW model is defined as the one central node and its nearest neighbors. The evolutions and distributions of node degree, strength and edge weight were mathematically analyzed. The predicted power-law behaviors consist with numerical simulations. The exponent functions of the power-law distributions in the NNLW model were found to be the same as the ones in the BBV model. It implies that the emergence of global preferential attachment could be from properly designed local connections.

Comparisons between those two models were made to investigate the influences of the nearest-neighbor connections on network evolution. When parameter $\delta$ is small, the NNLW model has a larger clustering coefficient and longer average shortest path length than those in the BBV model. The NNLW model also shows the hierarchical structure with the independence of parameter $\delta$. Besides, the NNLW model as a scale-free network is naturally ‘robust and yet fragile’, and its synchronization is generally weaker compared with the counterpart. All contrasts are added into our understanding of how the nearest-neighbor connections shape the structure and affect the performance of the weighed evolving network.

Acknowledgments

The authors thank reviewers for their constructive suggestions. This project was supported by the Jiangsu Planned Projects for Postdoctoral Research Funds (No. 1401014B).

References
