Lower bounds for the eigenvalues of Laplacian matrices

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Abstract

We give a lower bound for the second smallest eigenvalue of Laplacian matrices in terms of the isoperimetric number of weighted graphs. This is used to obtain an upper bound for the real parts of the nonmaximal eigenvalues of irreducible nonnegative matrices. © 2000 Elsevier Science Inc. All rights reserved.

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1. Introduction

The matrices in this paper are real and square. The eigenvalues of a matrix A will be ordered by

\[ \text{Re} \lambda_1(A) \leq \text{Re} \lambda_2(A) \leq \cdots \leq \text{Re} \lambda_n(A). \]

Given \( n \) real numbers \( x_1, \ldots, x_n \), \( \bar{x} \) will denote \( \max \{x_i\} \) and \( \underline{x} \) will denote \( \min \{x_i\} \).

Let \( C \) be an \( n \times n \) symmetric nonnegative irreducible matrix and let \( G \) be the graph of \( C \), i.e.,
V(G) = [1, 2, ..., n].

(i, j) ∈ E(G) iff i ≠ j and c_{ij} > 0.

The numbers c_{ij} can be considered as weights on the edges of G. So (G, C) is a weighted graph. Let δ_i be the i-th row sum of C, i = 1, ..., n, and let

\[ L_C(G) = \text{diag}[\delta_1, \ldots, \delta_n] - C \]

be the Laplacian matrix of (G, C) [9].

Let \( i_c(G) = \min(\sum_{i \in X, j \notin X} c_{ij} / |X|) \), where the minimum is taken over all non-empty subsets X of V(G) satisfying |X| ≤ \( \frac{1}{2} n \). We shall refer to \( i_c(G) \) as the isoperimetric number of the weighted graph (G, C). For C being a (0, 1) matrix, it is the isoperimetric number of G, \( i(G) = \min_{0 < |X| \leq n/2} |\partial X|/|X| \), where the edge boundary \( \partial X \) of X consists of all edges with exactly one edge point in X, e.g., [10,11]. This isoperimetric number is a discrete analogue of the Cheeger isoperimetric constant measuring the minimal possible ratio between the length (area) of a subset X of a Riemannian manifold M and the area (volume) of the smaller piece obtained by cutting M along X (see [1,4,11] and the references therein). The Laplacian matrix \( L_c(G) \) is a singular irreducible matrix (e.g. [9]). So

\[ 0 = \lambda_1(L_c(G)) < \lambda_2(L_c(G)). \]

In Section 2, we use \( i_c(G) \) to obtain a lower bound for \( \lambda_2(L_c(G)) \). This, in turn, will be applied to obtain, in Section 3, an upper bound for the real parts of the non-maximal eigenvalues of irreducible nonnegative matrices.

2. A lower bound for the second smallest eigenvalue of Laplacian matrices of weighted graphs

A matrix B is said to be diagonally symmetrizable if there exists a positive diagonal matrix D such that DB is symmetric. It is easy to see that in this case the eigenvalues of B are real. Let \( B[U] \) be the principal submatrix of B corresponding to a nonempty subset U of \{1, ..., n\}. The eigenvalues of \( B[U] \) are also real and from [7, proof of Lemma 1] it follows that:

**Lemma 2.1.** Let B be an \( n \times n \) diagonally symmetrizable irreducible matrix with nonpositive off-diagonal entries (Z-matrix). Then there exists a nonempty subset U of \{1, ..., n\} such that |U| ≤ n/2 and

\[ \lambda_1(B[U]) \leq \lambda_2(B). \]

(2.1)

The main result of the paper is as follows:
Theorem 2.2. Let \((G, C)\) be a weighted connected graph (i.e., \(C\) is irreducible) with \(n\) vertices. Let \(D = \text{diag}[d_1, \ldots, d_n]\) be a positive diagonal matrix and let \(\Omega = DL_c(G)\). Then
\[
\lambda_2(\Omega) \geq d \left( \delta - \sqrt{\delta^2 - i_c(G)^2} \right).
\]

Proof. By Lemma 2.1, there exists a nonempty subset \(U\) of \(V(G)\) such that \(|U| \leq n/2\) and
\[
\lambda_1(\Omega[U]) \leq \lambda_2(\Omega).
\]
Since \(L_c(G)\) is a singular \(M\)-matrix, \(\Omega[U]\) is an \(M\)-matrix; so it has a nonnegative eigenvector \(x\) corresponding to \(\lambda_1(\Omega[U])\) (e.g. [2]):
\[
\Omega[U]x = \lambda_1(\Omega[U])x, \quad x > 0.
\]
Multiplying by \(x^T D^{-1}[U]\) yields
\[
x^T L_c(G)[U]x = \lambda_1(\Omega[U])x^T D^{-1}[U]x.
\]
Define an \(n\)-dimensional vector \(f\) and a number \(\sigma\) by
\[
f_i = \begin{cases} x_i & \text{if } i \in U, \\ 0 & \text{if } i \notin U, \end{cases}
\]
and
\[
\sigma = \sum_{(i,j) \in E(G)} c_{ij} |f_i^2 - f_j^2|.
\]
Suppose \(f_i\) can get \(m + 1\) different values, say \(0 = t_0 < t_1 < \cdots < t_m\), for all \(i \in V(G)\). For \(k = 0, 1, \ldots, m\), let
\[
U_k = \{ i \in V(G): f_i \geq t_k \}
\]
and
\[
W_k = \{ i \in V(G): f_i = t_k \}.
\]
Then for \(k \geq 1\),
\[
|U_k| \leq |U| \leq \frac{1}{2}|V(G)|
\]
and
\[
U_k = W_k \cup W_{k+1} \cup \cdots \cup W_m.
\]
\[
V(G) - U_k = W_0 \cup W_1 \cup \cdots \cup W_{k-1}.
\]
Thus,
\[ \sigma = \sum_{k=1}^{m} \sum_{(i,j) \in E(G)} c_{ij} \left( f^2_i - f^2_j \right) \]

\[ = \sum_{k=1}^{m} \sum_{p=0}^{k-1} \sum_{(i,j) \in E(G)} c_{ij} \left( t_k^2 - t_p^2 \right) \]

\[ = \sum_{s=1}^{m} \sum_{i \in W_s, j \notin W_p} c_{ij} \left( t_s^2 - t_{s-1}^2 \right) \]

By the definition of the isoperimetric number \( i_c(G) \),

\[ \sum_{(i,j) \in E(G)} c_{ij} \geq i_c(G)|U_s|, \quad s = 1, 2, \ldots, m. \]

Hence,

\[ \sigma \geq \sum_{s=1}^{m} \left( t_s^2 - t_{s-1}^2 \right) |U_s| i_c(G) \]

\[ = i_c(G) \left\{ \sum_{s=1}^{m-1} t_s^2 |U_s| - |U_{s+1}| + t_m^2 |U_m| - |U_1| t_0^2 \right\} \]

\[ = \sum_{s=0}^{m} |W_s| t_s^2 i_c(G) = \sum_{i \in V(G)} f_i^2 i_c(G) \]

\[ = \sum_{i \in U} x_i^2 i_c(G) = i_c(G)x^T x. \quad (2.6) \]

By the Cauchy–Schwartz inequality,
\[ \sigma^2 = \left( \sum_{(i,j) \in E(G)} |f_i^2 - f_j^2| c_{ij} \right)^2 \]
\[ \leq \sum_{(i,j) \in E(G)} c_{ij} (f_i - f_j)^2 \sum_{(i,j) \in E(G)} c_{ij} (f_i + f_j)^2 \]
\[ = \left( f^T L_c(G) f \right) \left( f^T (2 \text{ diag} (\delta_1, \ldots, \delta_n) - L_c(G)) f \right) \]
\[ = \left( x^T L_c(G)[U] x \right) \left( x^T (2 \text{ diag} (\delta_1, \ldots, \delta_n)[U] - L_c(G)[U]) x \right). \]

By (2.4),
\[ \sigma^2 \leq \lambda_1(\Omega)[U]^T D^{-1}[U] x \left( \tilde{\delta}^2 x^T x - \lambda_1(\Omega)[U]^T D^{-1}[U] x \right). \]

Combining (2.6) and (2.7), we obtain
\[ \left( \lambda_1(\Omega)[U]^T D^{-1}[U] x \right)^2 - 2\tilde{\delta} x^T x \left( \lambda_1(\Omega)[U]^T D^{-1}[U] x \right) \]
\[ + (i_c(G)x^T x)^2 \leq 0. \]

Thus,
\[ \lambda_1(\Omega)[U]^T D^{-1}[U] x \geq \frac{2\tilde{\delta} x^T x - \sqrt{4\tilde{\delta}^2 (x^T x)^2 - 4i_c(G)^2 (x^T x)^2}}{2} \]
\[ = \left( \tilde{\delta} - \sqrt{\tilde{\delta}^2 - i_c(G)^2} \right) x^T x. \]

By (2.3),
\[ \lambda_1(\Omega)[U]^T D^{-1}[U] x \leq \frac{\lambda_2(\Omega)x^T x}{d} \]
and this inequality implies (2.2). \( \square \)

For simple graphs, (2.2) reduces to a result of [11].

**Corollary 2.3.** Let \( G \) be a simple connected graph with \( n \) vertices. Then
\[ \lambda_2(L(G)) \geq \tilde{\delta} - \sqrt{\tilde{\delta}^2 - i(G)^2}. \]  
(2.8)

**Proof.** The proof follows from Theorem 2.2, by choosing \( C \) as a \((0, 1)\)-matrix and \( D = I \). \( \square \)

**Corollary 2.4** [11]. Let \( G \) be a simple connected graph with \( n \geq 4 \) vertices. Then
\[ i(G) \leq \sqrt{\lambda_2(L(G))(2\tilde{\delta} - \lambda_2(L(G)))}. \]  
(2.9)
Proof. If $G \neq K_n$, then by [5], $\lambda_2(L(G)) \leq 2$. So by (2.2)\[ (\delta - \lambda_2(L(G)))^2 \leq \delta^2 - \iota(G)^2. \]
Hence, (2.9) holds. It also holds for $G = K_n$, $n \geq 4$, since in this case $\lambda_2(L(G)) = n$ and $i(G) = \lfloor n/2 \rfloor$ by [9,11]. □

3. Upper bound for the real parts of the nonmaximal eigenvalues of nonnegative irreducible matrices

For an $n \times n$ nonnegative matrix $A$ and for positive vectors $x$ and $y$ in $\mathbb{R}^n$ define\[ \varepsilon(A, x, y) = \min_{U, j \in U} \sum_{i \in U, j \notin U} (a_{ij}x_jy_i + a_{ji}x_jy_i), \]
where the minimum is taken over all nonempty subsets $U$ of $\{1, 2, \ldots, n\}$ with $|U| \leq n/2$. If $y = x$, we denote $\varepsilon(A, x, x)$ by $\varepsilon(A, x)$.

Lemma 3.1. Let $A$ be an $n \times n$ irreducible symmetric nonnegative matrix and let $u$ be a positive eigenvector of $A$, corresponding to the spectral radius, $\rho(A)$. Then\[ \lambda_{n-1}(A) \leq \sqrt{\rho(A)^2 - \frac{\varepsilon(A, u)^2}{u^2}}. \]

Proof. Let $G = G(A)$ and let $C = UAU$, where $U = \text{diag}[u_1, u_2, \ldots, u_n]$. Then $L_c(G) = U(\rho(A)I - A)U$ and choosing $D = U^{-2}$, $\Omega = U^{-1}(\rho(A)I - A)U$. Now, $\lambda_2(\Omega) = (\rho(A) - \lambda_{n-1}(A))$. Since $A$ is symmetric, $Au = \rho(A)u$, $u^T A = \rho(A)u^T$ and\[ i_c(G) = \min_{X \notin U} \frac{\sum_{i \in X, j \notin X} a_{ij}u_i u_j}{|X|} = \varepsilon(A, u). \]
So, by Theorem 2.2,\[ \rho(A) - \lambda_{n-1}(A) \geq \frac{\rho(A)u^2 - \sqrt{(\rho(A)u^2)^2 - \varepsilon(A, u)^2}}{u^2}. \]
Hence\[ \lambda_{n-1}(A) \leq \sqrt{\rho(A)^2 - \frac{\varepsilon(A, u)^2}{u^2}}. \]

Corollary 3.2. Let $G$ be a simple connected regular graph with degree $K$ on $n$ vertices and let $A$ be its adjacency matrix. Then\[ \lambda_{n-1}(A) \leq \sqrt{\rho(A)^2 - i(G)^2}. \]

Proof. Since $G$ is regular, $\rho(A) = K$ and $Ae = \rho(A)e$, where $e$ is a vector of ones. The claim follows from Lemma 3.1 and from the fact that $\varepsilon(A, e) = i(G)$. □
Theorem 3.3. Let \( A \) be an \( n \times n \) irreducible nonnegative matrix. Let \( Au = \rho(A)u \) and \( v^T A = \rho(A)v^T \), where \( u \) and \( v \) are positive vectors, \( w_i = u_i v_i \). Then

\[
\Re \lambda_{n-1}(A) \leq \sqrt{\rho(A)^2 - \frac{\varepsilon(A, u, v)^2}{\bar{\omega}^2}}.
\]

Proof. Let

\[
d_i = \sqrt{\frac{v_i}{u_i}}, \quad D = \text{diag}[d_1, \ldots, d_n],
\]

\[
B = \frac{DAD^{-1} + (DAD^{-1})^T}{2}.
\]

Then

\[
B(Du) = \frac{DAD^{-1}Du + D^{-1}A^T D^2 u}{2} = \frac{DAu + D^{-1}A^T D^2 u}{2} = \frac{D\rho(A)u + D^{-1}\rho(A)v}{2} = \frac{\rho(A)Du + Du\rho(A)}{2} = \rho(A)(Du).
\]

Moreover, a simple computation shows that \( \varepsilon(B, Du) = \varepsilon(A, u, v) \), and \( (Du)_i = \sqrt{u_i v_i} = \sqrt{w_i} \). On the other hand, it follows from [8, p. 237] that

\[
\rho(A) + \lambda_{n-1}(B) = \rho(B) + \lambda_{n-1}(B) \geq \rho(A) + \Re \lambda_{n-1}(A).
\]

Thus,

\[
\lambda_{n-1}(B) \geq \Re \lambda_{n-1}(A)
\]

and by Lemma 3.1,

\[
\Re \lambda_{n-1}(A) \leq \lambda_{n-1}(B) \leq \sqrt{\rho(B)^2 - \frac{\varepsilon(B, Du)^2}{\bar{\omega}^2}} = \sqrt{\rho(A)^2 - \frac{\varepsilon(A, u, v)^2}{\bar{\omega}^2}}. \quad \square
\]

Corollary 3.4. Let \( A \) be an \( n \times n \) irreducible doubly stochastic matrix and

\[
\mu(A) = \min \sum_{i \in U, j \in U} a_{ij} / |U|,
\]

where \( U \) is a union of \( \ell \) disjoint blocks of size \( r \). Then

\[
\Re \lambda_{n-1}(A) \leq \sqrt{\mu(A)^2 - \frac{\varepsilon(A, u, v)^2}{\bar{\omega}^2}}.
\]
where the minimum is taken over all nonempty subsets \( U \) of \( \{1, \ldots, n\} \) such that \( |U| \leq n/2 \). Then \( \lambda_{n-1}(A) \leq \sqrt{1 - \mu(A)^2} \).

**Proof.** The proof follows from Theorem 3.3, since for doubly stochastic matrices \( u = v = e \) and \( \varepsilon(A, e) = \mu(A) \). □

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**References**