\textit{$L^p$–$L^q$ decay estimates for Cauchy problems of linear thermoelastic systems with second sound in three dimensions}

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\textit{L}^p\textit{–}L^q \textit{decay estimates of solutions to Cauchy problems of linear thermoelastic systems with second sound in three spatial variables are studied. By carefully analysis of the different effects of higher, middle and lower frequencies in phase space, the asymptotic behaviour of characteristic roots of coefficient matrices is obtained. Then, with the help of the information on characteristic roots and the interpolation theorem, a decay estimate of parabolic type for the coupled system of the potential part of displacement, temperature and heat flux is obtained. Finally, \textit{L}^p\textit{–}L^q \textit{decay estimates of hyperbolic type for the original thermoelastic systems with second sound are obtained.}

1. Introduction

Thermoelastic systems describe the elastic and thermal behaviour of elastic heat-conductive media, particularly the reciprocal actions between elastic stresses and temperature differences [1–3, 6, 19–21]. Although the classical thermoelasticity theory, based on the Fourier law of heat conduction, predicts that a thermal disturbance of material conducting heat will be propagated with infinite speeds, experiments on certain dielectric crystals at very low temperature have shown the existence of the phenomenon of ‘second sound’, which means that the thermal disturbance is transmitted as ‘wave-like’ pulses with finite speeds (see [2, 3] and references therein). Mathematically, the effect of the second sound makes the thermoelastic systems purely hyperbolic. However, they are damped and based on the Cattaneo law for the heat conduction instead of the Fourier law (see [2, 3, 5, 7, 9, 10, 13] and references therein). There are some interesting works devoted to the thermoelastic systems with second sound (see [5, 7, 9, 10, 13] and references therein). For example, in [5] Gurtin and Pipkin developed a general theory of heat flow in rigid bodies with memory, which predicts finite propagation speeds for thermal disturbance. Tarabek [13] obtained the existence of smooth solutions of linear thermoelastic systems with second sound in the cases of both an unbounded body and a bounded body with pinned and insulated ends in one spatial variable by using the energy method.
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Racke [10] studied the long-time asymptotic behaviour of solutions of linear thermoelastic systems with second sound in two and three spatial variables. He proved an exponential stability for the pure, but lightly damped, hyperbolic system with Dirichlet-type boundary conditions. In [9], Racke proved that the dissipativity given through heat conduction is strong enough to exponentially stabilize all components \((u, \theta, q)\) in linear and nonlinear systems.

The \(L^p-L^q\) decay estimates can qualitatively describe the long-time behaviour of solutions. It is well known that the creation of an \(L^p-L^q\) decay estimate is a crucial step in studying the existence of global solutions with small initial data for nonlinear problems (cf. [6, 8] and references therein). There is a rich literature concerned with \(L^p-L^q\) decay estimates for thermoelastic systems of hyperbolic–parabolic coupled type (cf. [6, 8, 15] and references therein). Recently, the authors and their collaborators introduced a diagonalization method by frequency analysis in phase space in order to decouple the coupled systems, to obtain \(L^p-L^q\) decay estimates, and to study the propagation of singularity of solutions of linear and nonlinear problems (cf. [4, 11, 12, 14–19] and references therein). For instance, in [12] Reissig and Wang investigated the well-posedness, propagation of singularities and \(L^p-L^q\) decay estimates of solutions to Cauchy problems of linear thermoelastic systems of type III in one dimension by using the diagonalization argument in phase space. In [18] the authors found that solutions to the Cauchy problems of linear thermoelastic systems with second sound in one spatial variable have the same \(L^p-L^q\) decay rate as that in the heat equation.

The purpose of this paper is to study \(L^p-L^q\) decay estimates of solutions to the Cauchy problem for a linear thermoelastic system with second sound in three spatial variables. First, by carefully analysing the effects of lower, middle and high frequencies in phase space, respectively, an efficient method will be found to diagonalize the principal parts of the coupled system. Thus, the asymptotic behaviour of characteristic roots of the coefficient matrix may be obtained. With the help of the information of characteristic roots, \(L^p-L^q\) decay estimates of parabolic type for the coupled systems of the potential part of displacement, temperature and heat flux will be obtained. Finally, \(L^p-L^q\) decay estimates of hyperbolic type for the original thermoelastic systems with second sound are also obtained.

Consider the following Cauchy problem for a linear thermoelastic system with second sound in three spatial variables:

\[
\begin{cases}
y_{tt} + \alpha_1^2 \nabla \times \nabla \times y - \alpha_2^2 \nabla \nabla' y + \gamma \nabla \theta = 0, \\
\theta_t + \beta \nabla' q + \delta \nabla' y_t = 0, \\
\tau q_t + q + \kappa \nabla \theta = 0,
\end{cases}
\]

\[y(0, x) = y_0(x), \quad y_t(0, x) = y_1(x), \quad \theta(0, x) = \theta_0(x), \quad q(0, x) = q_0(x),\]  

(1.1)

where \(y\) and \(\theta\) are the displacement and the temperature difference of the elastic media, \(q\) is the heat flux, \(\alpha_1, \alpha_2, \beta, \delta, \gamma, \kappa, \tau\) are all positive constants with \(\alpha_1 < \alpha_2\). \(\Delta\) represents the Laplace operator, \(\nabla\) the gradient and \(\nabla'\) the divergence in spatial variables. The following notations are also used:

\[y_t = \frac{\partial y}{\partial t}, \quad y_{tt} = \frac{\partial^2 y}{\partial t^2}, \quad \theta_t = \frac{\partial \theta}{\partial t}, \quad q_t = \frac{\partial q}{\partial t}.\]
Decay estimates for Cauchy problems

In the following discussion, we shall decompose the vector $y$ as $y = y^p + y^s$ with $y^p$ being the potential part (i.e. $\nabla \times y^p = 0$) and $y^s$ being the solenoidal part (i.e. $\nabla \cdot y^s = 0$). From (1.1) we find that $y^s$ and $(y^p, \theta, q)$ satisfy the following problems:

$$
\begin{align*}
&y^s_{tt} - \alpha_1^2 \Delta y^s = 0, \\
&(y^s(0, x) = y^s_0(x), \quad y^s_t(0, x) = y^s_t(x),)
\end{align*}
$$

and

$$
\begin{align*}
&y^p_{tt} - \alpha_2^2 \nabla \cdot y^p + \gamma \nabla \theta = 0, \\
&\theta_t + \beta \nabla^i q + \delta \nabla^i y^p = 0, \\
&\tau q_t + q + \kappa \nabla \theta = 0, \\
&(y^p(0, x) = y^p_0(x), \quad y^p_t(0, x) = y^p_t(x), \quad \theta(0, x) = \theta_0(x), \quad q(0, x) = q_0(x).)
\end{align*}
$$

We denote by $W^{N,p}(R^3)$ the classical Sobolev space based on $L^p(R^3)$. The main results of this paper are as follows.

**Theorem 1.1.** If we let $(y^p, \theta, q)$ be the solution to the Cauchy problem (1.3), the following decay estimates of parabolic type hold:

$$
\| (y^p_0, \nabla y^p_0, \theta, q) \|_{L^\infty(R^3)} \leq C(1 + t)^{-3/2(1/p - 1/q)} \| (y^p_0, \nabla y^p_0, \theta_0, q_0) \|_{W^{N,p}(R^3)},
$$

where $N \geq 3(1 - 2q^{-1})$, $p^{-1} + q^{-1} = 1$, $2 \leq q \leq \infty$, and $C$ is a positive constant.

**Theorem 1.2.** Let $(y, \theta, q)$ be the unique solution to the Cauchy problem (1.1). It then satisfies the following $L^q - L^p$ decay estimates:

$$
\| (y_t, \nabla y_t, \theta, q) \|_{L^q(R^3)} \leq C(1 + t)^{-3(1/p - 1/q)} \| (y_0, \nabla y_0, \theta_0, q_0) \|_{W^{N,p}(R^3)},
$$

where $N \geq 3(1 - 2q^{-1})$, $p^{-1} + q^{-1} = 1$, $2 \leq q \leq \infty$, and $C$ is a positive constant.

From theorems 1.1 and 1.2, we can obtain the following result.

**Corollary 1.3.** Let $(y, \theta, q)$ be the unique solution to Cauchy problem (1.1) where the initial data satisfy $\text{rot } y_0 = \text{rot } y_1 = 0$. Then $(y, \theta, q)$ has the following decay estimates of parabolic type:

$$
\| (y_t, \nabla y_t, \theta, q) \|_{L^q(R^3)} \leq C(1 + t)^{-3/2(1/p - 1/q)} \| (y_0, \nabla y_0, \theta_0, q_0) \|_{W^{N,p}(R^3)},
$$

where $N \geq 3(1 - 2q^{-1})$, $p^{-1} + q^{-1} = 1$, $2 \leq q \leq \infty$, and $C$ is a positive constant.

The remainder of this paper is arranged as follows. In §2, we shall first transform the coupled system (1.3) into a system of order one. Its principal terms will then be diagonalized corresponding to the lower, higher and middle frequencies in phase space in order to obtain the asymptotic behaviour of characteristic roots of the coefficient matrix. Theorems 1.1 and 1.2 will be proved in §3.

2. Asymptotic behaviour of characteristic roots

The system (1.2) is a purely hyperbolic equation. Its well-posedness and $L^p-L^q$ decay estimates of solutions are well known (see [6,8] and references therein). For
example, the following $L^p$–$L^q$ decay estimates hold:

\[ \| (\nabla y_s^p, y_t^p) (t, \cdot) \|_{L^q (\mathbb{R}^3)} \leq C (1 + t)^{-\left( \frac{1}{p} - \frac{1}{q} \right)} \| (\nabla y_0^p, y_1^p) \|_{W^{N,p} (\mathbb{R}^3)}, \tag{2.1} \]

where $N \geq 3(1 - 2q^{-1})$, $p^{-1} + q^{-1} = 1$, $2 \leq q \leq \infty$ and $C$ is a positive constant.

It remains to investigate the coupled system (1.3) for the unknowns $(y^p, \theta, q)$ in the following procedure.

Denote by $\hat{y}^p$, $\hat{\theta}$ and $\hat{q}$ the Fourier transformations of $y^p$, $\theta$ and $q$, respectively, with respect to spatial variables $x \in \mathbb{R}^3$ and introduce

\[ \hat{y}^p_{\pm} = \hat{y}^p \pm i\alpha_2 |\xi| \hat{y}^p. \]

We find from (1.3) that

\[ \begin{aligned}
&\partial_t \hat{y}^p_{\pm} - i\alpha_2 |\xi| \hat{y}^p_{\pm} + i\gamma \xi \hat{\theta} = 0, \\
&\partial_t \hat{y}^p_{0} + i\alpha_2 |\xi| \hat{y}^p_{0} + i\gamma \xi \hat{\theta} = 0, \\
&\partial_t \hat{\theta} + i\beta \xi \cdot \hat{q} + i\delta \xi \cdot \frac{1}{2} (\hat{y}^p_{+} + \hat{y}^p_{-}) = 0, \\
&\partial_t \hat{q} + \frac{1}{\tau} \hat{q} + i\kappa \xi \frac{\hat{\theta}}{\tau} = 0.
\end{aligned} \tag{2.2} \]

Let $V = (\hat{y}^p_{+}, \hat{y}^p_{-}, \hat{\theta}, \hat{q})$. We know from (2.2) that $V$ satisfies

\[ \begin{aligned}
\partial_t V + A_0 V + A_1 V &= 0, \\
V(0, \xi) &= V_0(\xi),
\end{aligned} \tag{2.3} \]

where

\[ V_0(\xi) = (\hat{y}^p_{+} + i\alpha_2 |\xi| \hat{y}^p_{0}, \hat{y}^p_{-} - i\alpha_2 |\xi| \hat{y}^p_{0}, \hat{\theta}_0, \hat{\theta}_0)^T, \]

\[ A_0 = \text{diag} \left\{ 0, \ldots, 0, \frac{1}{\tau}, \frac{1}{\tau}, \frac{1}{\tau} \right\} \]

and

\[ A_1 = \begin{pmatrix}
-\alpha_2 |\xi| I_3 & 0_{3 \times 3} & i\gamma \xi & 0_{3 \times 3} \\
0_{3 \times 3} & \alpha_2 |\xi| I_3 & i\gamma \xi & 0_{3 \times 3} \\
\frac{1}{2} i\delta \xi^T & \frac{1}{2} i\delta \xi^T & 0 & i\beta \xi^T \\
0_{3 \times 3} & 0_{3 \times 3} & i\kappa \xi & 0_{3 \times 3}
\end{pmatrix}. \]

First, let $B := A_0 + A_1$. By a simple computation we have the following result.

**Lemma 2.1.** For any $\xi \in \mathbb{R}^3 \setminus \{0\}$, there are three eigenvalues of multiplicity 2, $\pm \alpha_2 |\xi|$ and 1/\tau, to the matrix $B$. Moreover, the left and right eigenvectors corresponding to $\pm \alpha_2 |\xi|$ are as follows:

\[ l_1 = r_1^T = (a_1, b_1, c_1, 0, \ldots, 0), \quad l_2 = r_2^T = (a_2, b_2, c_2, 0, \ldots, 0), \]

\[ l_3 = r_3^T = (0, 0, 0, a_1, b_1, c_1, 0, 0, 0, 0), \quad l_4 = r_4^T = (0, 0, a_2, b_2, c_2, 0, 0, 0, 0, 0), \]

where \( \{ (a_k, b_k, c_k) \}_{k=1}^4 \) are two unit vectors satisfying $a_1 \xi_1 + b_1 \xi_2 + c_1 \xi_3 = 0$ and $a_1 a_2 + b_1 b_2 + c_1 c_2 = 0.$
In the remainder of this section we shall carefully analyse the effects of frequencies in different regions of phase space to obtain the asymptotic behaviour of the eigenvalues of the matrix $\mathcal{B}$ and to find a method with which to diagonalize it.

### 2.1. In the region $|\xi| \leq \sigma \ll 1$

We restrict our discussion in this subsection to the region $|\xi| \leq \sigma$ for a small $\sigma > 0$.

Since the matrix $A_0$ is diagonal, we need only to diagonalize the matrix $A_1$ successively in the following procedure.

Let $V^1 = (I + K_1(\xi))V$, where $K_1$ is a zeroth-order matrix in $\xi$ and will be determined later. We find from (2.3) that

$$
\partial_t V^1 + A_0 V^1 + \left(\|\xi\|K_1, A_0\right) + A_1) V^1
+ \left(\|\xi\|^2[A_0, K_1[1 + |\xi|][K_1, A_1]]V^1 + A_1^2(\xi)V^1 = 0,
$$

where $[\cdot, \cdot]$ denotes the commutator of two related matrices and $A_1(\xi) = (C_{ij})_{10 \times 10}$ with $C_{ij} = O(|\xi|^3)$ when $|\xi| \to 0$.

Choose

$$
K_1 = i \begin{pmatrix}
0_{\sigma \times 7} & k_{12} \\
k_{21} & 0_{3 \times 3}
\end{pmatrix}
$$

with

$$
k_{12} = \left(0_{3 \times 6}, -\frac{\tau \beta \xi^T \gamma}{|\xi|}\right) \quad \text{and} \quad k_{12} = \left(0_{3 \times 6}, -\frac{\kappa \xi}{|\xi|}\right).
$$

By a direct computation we have

$$
A_1^1 := A_1 + |\xi|[K_1, A_0] = \begin{pmatrix}
-\text{i} \alpha_2 |\xi| I_3 & 0_{3 \times 3} & i \gamma \xi & 0_{3 \times 3} \\
0_{3 \times 3} & -\text{i} \alpha_2 |\xi| I_3 & 0_{3 \times 3} & i \gamma \xi \\
\frac{\tau \beta}{2} (i \delta \xi^T) & \frac{\tau \beta}{2} (i \delta \xi^T) & 0 & 0_{1 \times 3} \\
0_{3 \times 3} & 0_{3 \times 3} & 0_{3 \times 3} & 0_{3 \times 3}
\end{pmatrix}
$$

and

$$
A_1^2 = \begin{pmatrix}
0_{3 \times 3} & 0_{3 \times 3} & 0_{3 \times 1} & -\gamma \tau \beta (\xi_1 \xi_3)_{3 \times 3} \\
0_{3 \times 3} & 0_{3 \times 3} & 0_{3 \times 1} & -\gamma \tau \beta (\xi_1 \xi_3)_{3 \times 3} \\
0_{1 \times 3} & 0_{1 \times 3} & \beta \kappa |\xi|^2 & 0_{1 \times 3} \\
-\frac{1}{2} \kappa \delta (\xi_1 \xi_3)_{3 \times 3} & -\frac{1}{2} \kappa \delta (\xi_1 \xi_3)_{3 \times 3} & 0_{3 \times 1} & -\kappa \beta (\xi_1 \xi_3)_{3 \times 3}
\end{pmatrix},
$$

where $A_1^2 = [\xi][K_1, A_1] + |\xi|^2[A_0, K_1][K_1]$.

Thus, we obtain

$$
\partial_t V^1 + A_0^2 V^1 + A_1^1 V^1 + A_1^2 V^1 + A_1^2(\xi)V^1 = 0,
$$

where $A_0^1 = A_0$.

Let

$$
D = \begin{pmatrix}
-\text{i} \alpha_2 |\xi| I_3 & 0_{3 \times 3} & i \gamma \xi \\
0_{3 \times 3} & -\text{i} \alpha_2 |\xi| I_3 & 0_{3 \times 3} \\
\frac{1}{2} i \delta \xi^T & \frac{1}{2} i \delta \xi^T & 0
\end{pmatrix}
$$
be the left upper \((7 \times 7)\)-block of \(A^1_1\). We then find, from \(|\lambda I - D| = 0\), that
\[
\lambda_{1/2} = \ldots = O(|\xi|^3) \text{ as } |\xi| \to 0,
\]
\[
A^1_1 = \text{diag}\{-i\alpha_2|\xi|, -i\alpha_2|\xi|, i\alpha_2|\xi|, i\alpha_2|\xi|, 0, -im|\xi|, im|\xi|, 0, 0, 0\},
\]
and \(A^1_2\) is given in table 1.

Denote by \(l_k = (l_{1k}, l_{2k}, \ldots, l_{7k})\) and \(r_k = (r_{1k}, r_{2k}, \ldots, r_{7k})^T\) the left and right orthogonal eigenvectors of \(D\) with respect to the eigenvalues \(\lambda_k\) \((1 \leq k \leq 7)\). By a simple computation, we obtain
\[
l_1 = r_1^T = (a_1, b_1, c_1, 0, \ldots, 0),
\]
\[
l_2 = r_2^T = (a_2, b_2, c_2, 0, \ldots, 0),
\]
\[
l_3 = r_3^T = (0, 0, 0, a_1, b_1, c_1, 0),
\]
\[
l_4 = r_4^T = (0, 0, 0, a_2, b_2, c_2, 0),
\]
\[
l_5 = c_{15}(\delta \xi_1, \delta \xi_2, \delta \xi_3, -\delta \xi_1, -\delta \xi_2, -\delta \xi_3, 2|\alpha_2|),
\]
\[
r_5 = c_{15}(\gamma \xi_1, \gamma \xi_2, \gamma \xi_3, -\gamma \xi_1, -\gamma \xi_2, -\gamma \xi_3, 2|\alpha_2|)^T,
\]
\[
l_i = c_{ii}\left( \frac{i \delta \xi_1}{\lambda_i + i|\alpha_2|}, \frac{i \delta \xi_2}{\lambda_i + i|\alpha_2|}, \frac{i \delta \xi_3}{\lambda_i + i|\alpha_2|}, \frac{i \delta \xi_1}{\lambda_i - |\alpha_2|}, \frac{i \delta \xi_2}{\lambda_i - |\alpha_2|}, \frac{i \delta \xi_3}{\lambda_i - |\alpha_2|} \right),
\]
\[
r_i = c_{ri}\left( \frac{i \gamma \xi_1}{\lambda_i + i|\alpha_2|}, \frac{i \gamma \xi_2}{\lambda_i + i|\alpha_2|}, \frac{i \gamma \xi_3}{\lambda_i + i|\alpha_2|}, \frac{i \gamma \xi_1}{\lambda_i - |\alpha_2|}, \frac{i \gamma \xi_2}{\lambda_i - |\alpha_2|}, \frac{i \gamma \xi_3}{\lambda_i - |\alpha_2|} \right)^T,
\]
where \(i = 6, 7\), \(\{(a_k, b_k, c_k)\}_{k=1}^2\) are two unit vectors given as in lemma 2.1 and
\[
l_i r_j = \delta_{ij} \Rightarrow \begin{cases} C_{15} = \frac{1}{\sqrt{25^2 + 4|\alpha_2|^2}}, \\ C_{15} = \frac{1}{\sqrt{2\gamma^2 + |\alpha_2|^2}} \\ C_{15} C_{15} > 0, \quad 6 \leq i \leq 7. \end{cases}
\]

Let \(M_1 = (l_1, l_2, \ldots, l_7)^T\) and
\[
K_2 = \begin{pmatrix} M_1 & 0_{7 \times 3} \\ 0_{3 \times 7} & I_3 \end{pmatrix}.
\]

We then find that \(V^2 = K_2 V^1\) satisfies
\[
\partial_t V^2 + A^2_0 V^2 + A^2_1 V^2 + A^2_2 V^2 + A^2_3(\xi) V^2 = 0,
\]
where
\[
A^2_0 = A^1_1,
\]
\[
A^2_1(\xi) = (c_{ij})_{10 \times 10} \quad \text{with} \quad c_{ij} = O(|\xi|^3) \quad \text{as} \quad |\xi| \to 0,
\]
\[
A^2_1 = \text{diag}\{-i\alpha_2|\xi|, -i\alpha_2|\xi|, i\alpha_2|\xi|, i\alpha_2|\xi|, 0, -im|\xi|, im|\xi|, 0, 0, 0\},
\]
and \(A^2_3\) is given in table 1.
Note that \((\mathcal{C}_i, \mathcal{R}_i)\) \((i = 6, 7)\) are homogeneous of order zero in \(\xi\) and \((\mathcal{C}_{i6}, \mathcal{C}_{i7})\) are homogeneous of order \((-1)\). Thus, we know that all elements of \(A_2^2\) are of order \(O(|\xi|^2)\). Subsequently, we will diagonalize the matrix \(A_2^2\).

Let \(V^3 = (I + K_3|\xi|^2)V^2\), where \(K_3\) is a zeroth-order matrix in \(\xi\) and will be determined later. We find from (2.7) that
\[
\partial_t V^3 + A_2^3 V^3 + A_2^3 V^3 + (|\xi|^2[K_3, A_2^3] + A_2^3 V^3) + A_2^3(\xi)V^3 = 0,
\]
where \(A_2^3(\xi) = (C_{ij})_{10 \times 10}\) with \(C_{ij} = O(|\xi|^3)\) as \(|\xi| \to 0\).

Next we choose
\[
K_3 = \frac{1}{|\xi|^2} \begin{pmatrix}
0_{7 \times 5} & 0_{7 \times 2} & 0_{5 \times 3} \\
0_{3 \times 5} & x_{86} & x_{87} & y_{68} & y_{69} & y_{610} \\
x_{96} & x_{97} & x_{106} & x_{107} & y_{78} & y_{79} & y_{710} \\
0_{3 \times 3}
\end{pmatrix},
\]
where
\[
\begin{align*}
x_{86} &= i\lambda_6 C_{t77}\kappa_1, & x_{87} &= i\lambda_7 C_{t77}\kappa_1, & x_{96} &= i\lambda_6 C_{t67}\kappa_2, \\
x_{97} &= i\lambda_7 C_{t67}\kappa_2, & x_{106} &= i\lambda_6 C_{t67}\kappa_3, & x_{107} &= i\lambda_7 C_{t67}\kappa_3, \\
y_{68} &= -2i\lambda_6 C_{t67}\beta\tau^2\xi_1, & y_{69} &= -2i\lambda_6 C_{t67}\beta\tau^2\xi_2, & y_{610} &= -2i\lambda_6 C_{t67}\beta\tau^2\xi_3, \\
y_{78} &= -2i\lambda_7 C_{t77}\beta\tau^2\xi_1, & y_{79} &= -2i\lambda_7 C_{t77}\beta\tau^2\xi_2, & y_{710} &= -2i\lambda_7 C_{t77}\beta\tau^2\xi_3.
\end{align*}
\]
By a direct computation, we obtain
\[
\partial_t V^3 + A_2^3 V^3 + A_2^3 V^3 + A_2^3(\xi)V^3 = 0,
\]
where \(A_2^3 = A_5^3, A_1^3 = A_1^3,\) and \(A_2^3\) is given in table 1.

Let
\[
P = -\kappa\beta \begin{pmatrix}
\xi_1^2 & \xi_1\xi_2 & \xi_1\xi_3 \\
\xi_2\xi_1 & \xi_2^2 & \xi_2\xi_3 \\
\xi_3\xi_1 & \xi_3\xi_2 & \xi_3^2
\end{pmatrix}.
\]
From \(|\lambda I - P| = 0\), we have
\[
\lambda_1 = \lambda_2 = 0, \quad \lambda_3 = -\kappa\beta|\xi|^2.
\]
Owing to the real symmetry of matrix \(P\), we know that there is a orthogonal matrix \(M_2\) such that \(M_2PM_2^{-1} = \text{diag}\{-\kappa\beta|\xi|^2, 0, 0\}\).

Let
\[
K_4 = \begin{pmatrix}
I_{7 \times 7} & 0_{7 \times 3} \\
0_{3 \times 7} & M_2
\end{pmatrix}.
\]
Then we know from (2.9) that \(V^4 = K_4 V^3\) satisfies
\[
\partial_t V^4 + A_2^4 V^4 + A_2^4 V^4 + A_2^4(\xi)V^4 = 0,
\]
where \(A_2^4 = A_5^4, A_1^4 = A_1^4, A_2^4(\xi) = (C_{ij})_{10 \times 10}\) with \(C_{ij} = O(|\xi|^3)\) as \(|\xi| \to 0\), and \(A_2^4\) is given in table 1.

We shall diagonalize the matrix \(A_2^4\) in the following procedure.
Table 1. The matrices $A_2^2, A_3^2$ and $A_4^2$.

\[
A_2^2 = \begin{pmatrix}
0_{4\times 4} & 0_{4\times 3} & 0_{4\times 3} \\
0_{3\times 4} & \beta \kappa |\xi|^2 \begin{pmatrix}
2C_{i5}C_{r5}C_{r2}^2|\xi|^2 \\
2\alpha_2 C_{i5}C_{r6}|\xi| \\
2\alpha_2 C_{r6}C_{r7}|\xi| \\
2\alpha_2 C_{r7}C_{r7}|\xi|
\end{pmatrix} & 2\alpha_2 C_{i5}C_{r7}|\xi| \\
0_{3\times 4} & -i\kappa (0_{3\times 1} C_{r6}\lambda_6 \xi \ C_{r7}\lambda_7 \xi) & -\kappa\beta(\xi_\ell \xi_j)_{3\times 3} \\
0_{3\times 4} & 0_{3\times 3} & 0_{3\times 3}
\end{pmatrix}
\]

\[
A_3^2 = \begin{pmatrix}
0_{4\times 4} & 0_{4\times 3} & 0_{4\times 3} \\
0_{3\times 4} & \beta \kappa |\xi|^2 \begin{pmatrix}
2C_{i5}C_{r5}C_{r2}^2|\xi|^2 \\
2\alpha_2 C_{r6}C_{r2}^2|\xi| \\
2\alpha_2 C_{r6}C_{r7}|\xi| \\
2\alpha_2 C_{r7}C_{r7}|\xi|
\end{pmatrix} & 2\alpha_2 C_{i5}C_{r7}|\xi| \\
0_{3\times 4} & 0_{3\times 3} & -\kappa\beta(\xi_\ell \xi_j)_{3\times 3} \\
0_{3\times 4} & 0_{3\times 3} & 0_{3\times 3}
\end{pmatrix}
\]

\[
A_4^2 = \beta \kappa |\xi|^2 \begin{pmatrix}
0_{4\times 4} & 0_{4\times 3} & 0_{4\times 3} \\
0_{3\times 4} & \begin{pmatrix}
2C_{i5}C_{r5}C_{r2}^2|\xi|^2 \\
2\alpha_2 C_{r6}C_{r2}^2|\xi| \\
2\alpha_2 C_{r6}C_{r7}|\xi| \\
2\alpha_2 C_{r7}C_{r7}|\xi|
\end{pmatrix} & 2\alpha_2 C_{i5}C_{r7}|\xi| \\
0_{3\times 4} & 0_{3\times 3} & 0_{3\times 3} \\
0_{4\times 4} & 0_{3\times 3} & \text{diag}\{-1,0,0\}
\end{pmatrix}
\]
Let $V^5 = (I + K_5|\xi|)V^4$, where $K_4$ is a zeroth-order matrix in $\xi$ and will be determined later. We then see from (2.10) that

$$\partial_\tau V^5 + A_5^4 V^5 + (A_4^4 + |\xi|[K_5, A_4^4])V^5 + (|\xi|[K_5, A_4^4] + A_2^4 - |\xi|^2[K_5, A_6^4]K_5)V^5 + A_5^5(\xi)V^5 = 0,$$

(2.11)

where $A_5^5(\xi) = (C_{ij})_{10 \times 10}$ with $C_{ij} = O(|\xi|^3)$ when $|\xi| \to 0$.

If we choose

$$K_5 = \begin{pmatrix}
0_{4 \times 4} & 0_{4 \times 3} & 0_{4 \times 3} \\
0_{3 \times 4} & x_{56} & x_{57} \\
0_{3 \times 4} & x_{65} & x_{67} \\
0_{3 \times 4} & x_{75} & x_{76} \\
0_{3 \times 3} & 0_{3 \times 3} & 0_{3 \times 3}
\end{pmatrix},$$

with

$$x_{56} = -\frac{2C_{15}C_{16}\alpha_2\beta_2|\xi|^2}{m}, \quad x_{57} = \frac{2C_{15}C_{17}\alpha_2\beta_3|\xi|^2}{m}, \quad x_{65} = \frac{2C_{16}C_{17}\alpha_2\beta_3|\xi|^2}{m},$$

and take $m$ as given in (2.6), we find by a direct computation that

$$[K_5, A_6^4] = 0$$

and $A_5^5 = |\xi|[K_5, A_4^4] + A_4^4$ is diagonal:

$$A_6^5 = \text{diag} \{0, 0, 0, 0, 2C_{15}C_{16}\alpha_2\beta_2|\xi|^4, 2C_{15}C_{17}\alpha_2\beta_3|\xi|^2, 2C_{17}C_{17}\beta_3|\xi|^2, \beta_2|\xi|^2, 2C_{17}C_{17}\beta_3|\xi|^2, 0, 0, 0\}.$$

Thus, from (2.11), we have

$$\partial_\tau V^5 + A_5^4 V^5 + A_4^4 V^5 + A_2^4 V^5 + A_5^5(\xi)V^5 = 0,$$

(2.12)

where $A_5^5 = A_3^5$ and $A_4^4 = A_4^4$.

Noting the diagonalization procedure from (2.4)–(2.11) and lemma 2.1, we know that

$$A_5^5(\xi) = \begin{pmatrix}
0_{4 \times 4} & 0_{4 \times 3} & 0_{4 \times 3} \\
0_{3 \times 4} & x_{56} & x_{57} \\
0_{3 \times 4} & x_{65} & x_{67} \\
0_{3 \times 4} & x_{75} & x_{76} \\
0_{3 \times 3} & 0_{3 \times 3} & 0_{3 \times 3}
\end{pmatrix},$$

and

$$C_{ij} = O(|\xi|^3)$$

as $|\xi| \to 0$.

Then, from (2.12), we obtain the following lemma.

**Lemma 2.2.** There is a small constant $\sigma > 0$ such that, when $|\xi| \leq \sigma$, the characteristic roots $\nu_k = \nu_k(\xi)$ $(1 \leq k \leq 10)$ of the matrix $B$ behave as

$$\nu_1 = \nu_2 = -i|\xi|\alpha_2, \quad \nu_3 = \nu_4 = i|\xi|\alpha_2, \quad \nu_5 = 2C_{15}C_{15}\alpha_2^2\beta_2|\xi|^4 + O(|\xi|^3),$$

$$\nu_6 = im|\xi| + 2C_{16}C_{16}\beta_2|\xi|^2 + O(|\xi|^3), \quad \nu_7 = -im|\xi| + 2C_{17}C_{17}\beta_3|\xi|^2 + O(|\xi|^3),$$

$$\nu_8 = -\kappa_2|\xi|^2 + \frac{1}{\tau} + O(|\xi|^3), \quad \nu_9 = \nu_{10} = \frac{1}{\tau} + O(|\xi|^3),$$

(2.13)

where $C_{15}C_{15} \geq C|\xi|^2$ and $C_{16}C_{16} \geq C$ $(i = 6, 7)$, for a positive constant $C > 0$. 


Proposition 2.3. Let \( V(t, \xi) = (V_1, V_2, \ldots, V_{10})^T \) be the solution to the Cauchy problem
\[
\partial_t V + A_0 V + A_1 V = 0, \quad V(0, \xi) = V_0(\xi)
\]
and let \( F^{-1}(V^i(t, \xi)) \) (\( i = 1, 2 \)), where
\[
V^1(t, \xi) = (V_1, V_2, V_3)^T \quad \text{and} \quad V^2(t, \xi) = (V_4, V_5, V_6)^T,
\]
is rotation-free for all \( t \geq 0 \). We then have, with \( |\xi| \leq \sigma \ll 1 \), the following representation:
\[
V(t, \xi) = \sum_{r=5}^{10} \sum_{l=1}^{10} \exp(-\nu_l(\xi)t)C_{rlk}(\xi) V_{l0}(\xi),
\]
where the \( C_{rlk}(\xi) \) tend to constants \( C_{rlk}^0 \) as \( |\xi| \to 0 \) and the \( C_{rlk}(\xi) \) are bounded on \( |\xi| \leq \sigma \).

Proof. From lemmas 2.1 and 2.2, we know that there is an invertible matrix \( L(\xi) \) such that \( D := L(\xi)BL^{-1}(\xi) = \text{diag}\{\nu_1, \ldots, \nu_{10}\} \), where \( B := A_0 + A_1 \). Moreover, we may choose \( L(\xi) = (l_1, \ldots, l_{10})^T \) such that \( l_i \) (\( 1 \leq i \leq 4 \)) are the linear independent left eigenvectors corresponding to \( \nu_i \) (\( 1 \leq i \leq 4 \)), respectively, and which satisfy \( l_i \cdot l_j = 0 \) (\( i \neq j \)). Thus, from the rotation-free property of \( F^{-1}(V^i) \) (\( i = 1, 2 \)), we have
\[
l_i \cdot V_0(\xi) = 0, \quad 1 \leq i \leq 4.
\]
Let \( W = LV = (W_1, W_2, \ldots, W_{10})^T \). We then see from (2.15) that \( W \) satisfies
\[
\begin{align*}
&\partial_t W + DW = 0, \\
&W_0 = (0, 0, 0, 0, W_5(0), \ldots, W_{10}(0))^T,
\end{align*}
\]
where \( W_5(0) \) (\( 5 \leq i \leq 10 \)) is from \( L(\xi)V_0(\xi) \).

We obtain, by a direct computation from (2.16),
\[
\begin{align*}
&W_i(t, \xi) = 0, \quad i = 1, 2, 3, 4, \\
&W_j(t, \xi) = \exp(-\nu_j t)W_j(0), \quad 5 \leq j \leq 10.
\end{align*}
\]

Using the backward transformation from \( W \) to \( V \), we see from (2.17) that proposition 2.3 holds.

2.2. In the region \( |\xi| \geq N \gg 1 \)

In this subsection, we will restrict our discussion to the region \( |\xi| \geq N \gg 1 \).

First, we shall diagonalize the main part \( A_1 \) of (2.3) in the following procedure.

By a direct computation, we find from \( |\lambda I - A_1| = 0 \) that
\[
\lambda^2(\lambda + i\alpha_2|\xi|)^2(\lambda - i\alpha_2|\xi|)^2 \left[ \lambda^4 + \left( \delta_\gamma + \alpha_2^2 + \frac{\beta_\kappa}{\tau} \right)|\xi|^2 \lambda^2 + \frac{\alpha_2^2\beta_\kappa|\xi|^4}{\tau} \right] = 0.
\]

As a consequence of (2.18), we have
\[
\begin{align*}
&\lambda_{1/2} = -i\alpha_2|\xi|, \quad \lambda_{3/4} = i\alpha_2|\xi|, \quad \lambda_{5/6} = 0, \\
&\lambda_{7/8} = \pm i|\xi|\sqrt{\frac{d + c}{2}}, \quad \lambda_{9/10} = \pm i|\xi|\sqrt{\frac{d - c}{2}}.
\end{align*}
\]
Decay estimates for Cauchy problems

with
d = \delta \gamma + \alpha^2 + \beta \kappa \tau and c = \sqrt{\left(\frac{\delta \gamma + \alpha^2}{\tau} + \beta \kappa \tau \right)^2 - 4\alpha^2 \beta \kappa \tau}.

Obviously, we see from (2.19) that

\lambda^2 < 0 \quad \text{as} \quad \lambda \neq 0. \quad (2.20)

Denote by \( l_i \) and \( r_i \) the left and right orthogonal eigenvectors of the matrix \( A_1 \) with respect to \( \lambda_i \) (\( 1 \leq i \leq 10 \)). By a direct computation, we have

\[
\begin{align*}
l_1 &= r_1^T = (a_1, b_1, c_1, 0, \ldots, 0), \\
l_2 &= r_2^T = (a_2, b_2, c_2, 0, \ldots, 0), \\
l_3 &= r_3^T = (0, 0, 0, a_1, b_1, c_1, 0, 0, 0, 0), \\
l_4 &= r_4^T = (0, 0, a_2, b_2, c_2, 0, 0, 0), \\
l_5 &= r_5^T = (0, 0, 0, 0, 0, 0, a_1, b_1, c_1), \\
l_6 &= r_6^T = (0, 0, 0, 0, 0, a_2, b_2, c_2),
\end{align*}
\]

and

\[
\begin{align*}
l_i &= C_{l_i} \left( \frac{i\delta \xi_1}{\lambda_i + i\alpha_2|\xi|}, \ldots, \frac{i\delta \xi_1}{\lambda_i + i\alpha_2|\xi|}, \frac{2i\beta \xi_3}{\lambda_i} \right), \\
r_i &= C_{r_i} \left( \frac{i\gamma \xi_1}{\lambda_i + i\alpha_2|\xi|}, \ldots, \frac{i\gamma \xi_1}{\lambda_i + i\alpha_2|\xi|}, \frac{i\lambda \xi_3}{\lambda_i \tau} \right),
\end{align*}
\]

where \( 7 \leq i \leq 10 \), \( \{(a_k, b_k, c_k)\}_{k=1}^7 \) are two unit vectors as given in lemma 2.1 and

\[
l_i r_j = \delta_{ij} \Rightarrow C_{l_i} C_{r_i} > 0, \quad 7 \leq i \leq 10. \quad (2.21)
\]

Let \( L_1 = (l_1, l_2, \ldots, l_{10})^T \) and \( V^1 = L_1 V \). By a direct computation, we find from (2.3) that

\[
\partial_t V^1 + A_1^1 V^1 + A_0^1 V^1 = 0, \quad (2.22)
\]

where

\[
A_1^1 = \text{diag}\{\lambda_1, \lambda_2, \lambda_3, \lambda_4, 0, 0, \lambda_7, \ldots, \lambda_{10}\}
\]

and

\[
A_0^1 = \begin{pmatrix}
0_{4 \times 4} & 0_{4 \times 6} \\
0_{6 \times 4} & \begin{pmatrix}
\frac{1}{\tau} & 0 & 0 & \cdots & 0 \\
0 & \frac{1}{\tau} & 0 & \cdots & 0 \\
0 & 0 & -\frac{2C_{\tau}C_{\tau} \beta \kappa |\xi|^2}{\tau^2 \lambda_7^2} & \cdots & -\frac{2C_{\tau}C_{\tau} \beta \kappa |\xi|^2}{\tau^2 \lambda_7 \lambda_{10}} \\
0 & 0 & -\frac{2C_{\tau}C_{\tau} \beta \kappa |\xi|^2}{\tau^2 \lambda_7 \lambda_8} & \cdots & \vdots \\
0 & 0 & -\frac{2C_{\tau}C_{\tau} \beta \kappa |\xi|^2}{\tau^2 \lambda_7 \lambda_{10}} & \cdots & -\frac{2C_{\tau}C_{\tau} \beta \kappa |\xi|^2}{\tau^2 \lambda_7 \lambda_{10}} \\
\end{pmatrix}
\end{pmatrix}
\]
In the following procedure we will diagonalize the matrix $A_0^1$.

Let

$$L_2 = \begin{pmatrix} 
0_{6 \times 6} & 0_{6 \times 4} \\
0_{4 \times 6} & \begin{pmatrix}
0 & x_{78} & x_{79} & x_{710} \\
x_{87} & 0 & x_{89} & x_{810} \\
\vdots & \vdots & \vdots & \vdots \\
x_{107} & x_{108} & x_{109} & 0
\end{pmatrix}
\end{pmatrix},$$

where $x_{ij}$ ($7 \leq i, j \leq 10$) are zeroth order in $\xi$ and will be determined later.

We then see that $V^2 = (I + |\xi|^{-1}L_2)V^1$ satisfies

$$\partial_t V^2 + A_1^1 V^2 + (|\xi|^{-1}[L_2, A_1^1] + A_0^1)V^2 + A_{21}^2(\xi)V^2 = 0,$$

(2.23)

where, in particular, by the constitution of $L_2$ and lemma 2.1, we find that

$$A_{21}^2(\xi) = \begin{pmatrix} 0_{4 \times 10} \\
(C_{ij})_{6 \times 6}
\end{pmatrix}$$

with $C_{ij} = O(|\xi|^{-1})$ as $|\xi| \to \infty$.

By a direct computation, we have

$$[L_2, A_1^1] = \begin{pmatrix} 0_{6 \times 6} & 0_{6 \times 4} \\
0_{4 \times 6} & ((\lambda_j - \lambda_i)x_{ij})_{4 \times 4}
\end{pmatrix},$$

where $\lambda_k (7 \leq k \leq 10)$ from (2.19).

Obviously, owing to the distinction of $\lambda_i$ ($7 \leq i \leq 10$), by an appropriate choice of $x_{ij}$ ($7 \leq i, j \leq 10$) we can obtain

$$\partial_t V^2 + A_1^1 V^2 + A_0^1 V^2 + A_{21}^2(\xi)V^2 = 0,$$

(2.24)

where $A_1^2 = A_1^1$ and

$$A_0^2 = \begin{pmatrix} 0_{4 \times 4} & 0_{4 \times 6} \\
0_{6 \times 4} & E_{6 \times 6}
\end{pmatrix}$$

with

$$E_{6 \times 6} = \text{diag} \left\{ 1, 1, \frac{4C_{77}C_{78}\beta\kappa}{\tau^2(d + c)}, \frac{4C_{17}C_{18}\beta\kappa}{\tau^2(d + c)}, \frac{4C_{97}C_{98}\beta\kappa}{\tau^2(d - c)}, \frac{4C_{107}C_{108}\beta\kappa}{\tau^2(d - c)} \right\}.$$

By using the principle of similarity of matrix, we obtain, from (2.24) and lemma 2.1, the following lemma.

**Lemma 2.4.** The characteristic roots $\nu_k$ ($1 \leq k \leq 10$) of the matrix $B = A_1 + A_0$ behave for $|\xi| \geq N \gg 1$ as

$$\nu_{1/2} = -i\alpha_2|\xi|, \quad \nu_{3/4} = i\alpha_2|\xi|, \quad \nu_5 = \frac{1}{\tau} + O(|\xi|^{-1}), \quad \nu_6 = \frac{1}{\tau} + O(|\xi|^{-1}),$$

$$\nu_7 = \frac{4C_{77}C_{78}\beta\kappa}{\tau^2(d + c)} + O(|\xi|^{-1}), \quad \nu_8 = \frac{4C_{17}C_{18}\beta\kappa}{\tau^2(d + c)} + O(|\xi|^{-1}),$$

$$\nu_9 = \frac{4C_{97}C_{98}\beta\kappa}{\tau^2(d - c)} + O(|\xi|^{-1}), \quad \nu_{10} = \frac{4C_{107}C_{108}\beta\kappa}{\tau^2(d - c)} + O(|\xi|^{-1}).$$

(2.25)
Using lemma 2.4, by the Duhamel principle we can obtain the following result, which is similar to proposition 2.3.

**Proposition 2.5.** Let $V(t, \xi) = (V_1, V_2, \ldots, V_{10})^T$ be the solution to the Cauchy problem

$$\partial_t V + A_0 V + A_1 V = 0, \quad V(0, \xi) = V_0(\xi)$$

and $F^{-1}(V^i(t, \xi))$ \((i = 1, 2)\), where

$$V^1(t, \xi) = (V_1, V_2, V_3)^T \quad \text{and} \quad V^2(t, \xi) = (V_4, V_5, V_6)^T,$$

is rotation-free for all $t \geq 0$. Then we have, by $|\xi| \geq N \gg 1$, the following representation:

$$V(t, \xi) = \sum_{r=5}^{10} \sum_{l=1}^{10} \exp(-\nu_r(\xi)t)C_{rlk}(\xi)V_{l,0}(\xi),$$

where the $C_{rlk}(\xi)$ tend to constants $C_{rlk}^0$ as $|\xi| \to \infty$, and the $C_{rlk}(\xi)$ are bounded on $|\xi| \geq N \gg 1$.

The proof of proposition 2.5 is similar to that of proposition 2.3. We omit it here.

### 2.3. In the region $\sigma \leq |\xi| \leq N$

In this subsection, we will restrict our discussion to the region $\sigma \leq |\xi| \leq N$. Let $\nu_k$ \((1 \leq k \leq 10)\) denote the characteristic roots of $B = A_1 + A_0$ in (2.3). From lemmas 2.2 and 2.4, and lemma 2.6 below, we find, with the help of compactness of \(\{\sigma \leq |\xi| \leq N\}\), that

$$\text{Re}\ \nu_i = 0, \quad 1 \leq i \leq 4, \quad \text{and} \quad \text{Re}\ \nu_k(\xi) \geq C > 0, \quad 5 \leq k \leq 10,$$

for all $\xi \in \{\sigma \leq |\xi| \leq N\}$, where $C$ represents a constant.

**Lemma 2.6.** The matrix $B = A_1 + A_0$ in (2.3) has no other purely imaginary eigenvalues $\nu = ia$, $a \in \mathbb{R}$, $a \neq 0$, in $\{\sigma \leq |\xi| \leq N\}$ besides $\nu_{1/2} = -i\alpha_2|\xi|$ and $\nu_{3/4} = i\alpha_2|\xi|$.

**Proof.** Suppose that $\nu = ia$, $a \neq 0$, $a \in \mathbb{R}$ is one of the eigenvalues of $B$ and is distinct from $\pm i\alpha_2|\xi|$ for any $|\xi| \neq 0$.

Let

$$|\lambda I - B| = 0.$$  \hspace{1cm} (2.29)

By a direct computation, we obtain from (2.29) that

$$(\lambda + i\alpha_2|\xi|)^2(\lambda - i\alpha_2|\xi|)^2 \left(\lambda - \frac{1}{\tau}\right)^2$$

$$\times \left[\lambda^4 - \frac{1}{\tau}\lambda^3 + \left(\alpha_2^2 + \gamma\delta + \frac{\kappa\beta}{\tau}\right)|\xi|^2\lambda^2 - \frac{(\alpha_2^2 + \gamma\delta)|\xi|^2}{\tau}\lambda + \frac{\alpha_2^2\kappa\beta|\xi|^4}{\tau}\right] = 0.$$  \hspace{1cm} (2.30)
In terms of the previous assumption, we know from (2.30) that $ia$ satisfies
\[(a + \alpha^2|\xi|^2)(ia - \frac{1}{\tau})^2\times\frac{a - \alpha^2|\xi|}{\tau}^2 = 0 \quad (2.31)\]

Consequently, we can deduce from (2.31) (and by considering the hypothesis on $\nu = ia$) that
\[
\begin{align*}
\frac{a^3}{\tau} - (\frac{\alpha^2}{\tau} + \gamma\delta)|\xi|^2 a^2 = 0, \\
a^4 - (\frac{\alpha^2}{\tau} + \gamma\delta + \frac{\kappa\beta}{\tau})|\xi|^2 a^2 + \frac{\alpha^2\kappa\beta|\xi|^4}{\tau} = 0.
\end{align*} \quad (2.32)
\]

Obviously, the solutions of the first equation in (2.32), considering $a \neq 0$, are
\[
a^2 = (\frac{\alpha^2}{\tau} + \gamma\delta)|\xi|^2. \quad (2.33)
\]
By substituting (2.33) in the second equation of (2.32), we obtain
\[
a^2 = \alpha^2|\xi|^2. \quad (2.34)
\]
From (2.33) and (2.34), we obtain
\[
\gamma\delta = 0, \quad (2.35)
\]
which contradicts the previous assumptions in (1.1). Thus, our hypothesis at the beginning of this proof is wrong and lemma 2.6 holds.

**Proposition 2.7.** Let $V(t, \xi) = (V_1, V_2, \ldots, V_{10})^T$ be the solution to the Cauchy problem
\[
\partial_t V + A_0 V + A_1 V = 0, \quad V(0, \xi) = V_0(\xi),
\]
and $F^{-1}(V(t, \xi))$ $(i = 1, 2)$, where
\[
V^1(t, \xi) = (V_1, V_2, V_3)^T \quad \text{and} \quad V^2(t, \xi) = (V_4, V_5, V_6)^T,
\]
is rotation-free for all $t \geq 0$. There then exist two positive constants $C_1$ and $C_2$ such that, for $\sigma \leq |\xi| \leq N$, we have the following decay estimates:
\[
|V(t, \xi)| \leq C_1 \exp(-C_2t)|V_0(\xi)|, \quad (2.36)
\]
where $C_2 \in (0, C)$, with $C > 0$ being given in (2.28).

**Proof.** From (2.28), we know that $\text{Re} \nu_k = 0$ $(1 \leq k \leq 4)$ and $\text{Re} \nu_k \geq C > 0$ $(5 \leq k \leq 10)$ for all $\xi \in \{\sigma \leq |\xi| \leq N\}$. Let us consider the equation $\partial_t V + B V = 0$ with $B = A_1 + A_0$. From lemma 2.1 we know that there exists an invertible matrix $L = L(\xi)$ such that
\[
LBL^{-1} := A = \begin{pmatrix} H_1 & 0 \\ 0 & H_2 \end{pmatrix},
\]
where \( H_1 = \text{diag}\{\nu_1, \nu_2, \nu_3, \nu_4\} \) and \( H_2 \) is a Jordan matrix of order six which depends on the multiplicity of eigenvalues \( \nu_i \) (5 \( \leq \) \( i \) \( \leq \) 10) of the matrix \( B \). As in the proof of proposition 2.3, we may choose \( L(\xi) = (l_1, l_2, \ldots, l_{10})^T \), such that \( l_i \cdot V_0(\xi) = 0 \) (\( i = 1, 2, 3, 4 \)).

Thus, if we let \( W = LV = (W_1, W_2, \ldots, W_9, W_{10})^T \), we then see, for all \( \xi \in \{ \sigma \leq |\xi| \leq N \} \), that \( W \) satisfies

\[
\begin{align*}
\partial_t W + AW &= 0, \\
W(0, \xi) &= (0, 0, 0, 0, W_5(0), \ldots, W_{10}(0))^T,
\end{align*}
\]  

(2.37)

where \( W_i(0) \) (\( i = 5, \ldots, 10 \)) is from \( L(\xi)V_0(\xi) \).

By a direct computation, we have

\[
W_i = 0, \quad i = 1, 2, 3, 4.
\]  

(2.38)

Thus, we are only interested in \( W_j \) (\( j = 5, \ldots, 10 \)) in the following procedure.

If all roots \( \nu_i \) (5 \( \leq \) \( i \) \( \leq \) 10) are simple, we can obtain from (2.28) and (2.37) that

\[
|W_j(\xi)| \leq C_1 \exp(-C_2 t)|V_0(\xi)|, \quad 5 \leq j \leq 10.
\]  

(2.39)

Otherwise, let us consider, for example, the case that the characteristic roots satisfy \( \nu_5 = \nu_6, \nu_7 = \nu_8 \) and \( \nu_9 = \nu_{10} \), but that \( \nu_5 \neq \nu_7 \neq \nu_9 \) at a point \( \xi_0 \in \{ \sigma \leq |\xi| \leq N \} \). By a direct computation, we have

\[
H_2 = \begin{pmatrix}
P_5(\xi_0) & 0_{2 \times 2} & 0_{2 \times 2} \\
0_{2 \times 2} & P_7(\xi_0) & 0_{2 \times 2} \\
0_{2 \times 2} & 0_{2 \times 2} & P_9(\xi_0)
\end{pmatrix},
\]  

(2.40)

where

\[
P_j(\xi_0) = \begin{pmatrix}
\nu_j(\xi_0) & 0 \\
1 & \nu_j(\xi_0)
\end{pmatrix}, \quad j = 5, 7, 9.
\]

From (2.37) and (2.40), we can obtain

\[
\begin{align*}
\partial_t W_5 + \nu_5(\xi_0)W_5 &= 0, \\
\partial_t W_6 + \nu_5(\xi_0)W_6 + W_5 &= 0, \\
\vdots & \\
\partial_t W_9 + \nu_9(\xi_0)W_9 &= 0, \\
\partial_t W_{10} + \nu_9(\xi_0)W_{10} + W_9 &= 0,
\end{align*}
\]  

(2.41)

with the initial conditions

\[
W_k(0, \xi_0) = e_k(\xi_0) = \sum_{p=1}^{10} l_{kp}(\xi_0)V_{0,p}(\xi_0), \quad 5 \leq k \leq 10,
\]  

(2.42)

where \( l_{kp} \) represents the \( p \)-th component of the \( k \)-th row of the matrix \( L \) and \( V_{0,p} \) represents the \( p \)-th component of the vector \( V_0(\xi_0) \).

By a direct computation, it follows from (2.41) and (2.42) that

\[
W_i(t, \xi_0) = \exp(-\nu_i(\xi_0)t)e_i(\xi_0), \quad i = 5, 7, 9.
\]  

(2.43)
From (2.41)–(2.43), we can easily obtain
\[ W_j(t, \xi_0) = \exp(-\nu_j(\xi_0)t)(e_j(\xi_0) - e_{j-1}(\xi_0)t), \quad j = 6, 8, 10. \] (2.44)

Considering (2.28), we find from (2.43) and (2.44) that
\[ |W_j(t, \xi_0)| \leq C_1 \exp(-C_2t)|V_0(\xi_0)|, \] (2.45)
where \( C_2 \in (0, C) \) (\( C \) is the constant from (2.28)).

For any \( \xi_0 \in \{ \sigma \leq |\xi| \leq N \} \), we can derive (2.45) by using the transformation to a Jordan form as in the previous procedure. It now remains to verify (2.45) for any \( \xi \in \{ \sigma \leq |\xi| \leq N \} \) with the help of the compactness of \( \{ \sigma \leq |\xi| \leq N \} \).

Let us consider
\[ \partial_t Z + H_2(\xi)Z = 0, \] (2.46)
where \( Z := (W_5, \ldots, W_{10})^T \) and \( \xi \) is in a small neighbourhood of \( \xi_0 \).

Obviously, (2.46) can be rewritten as
\[ \partial_t \chi(t, s, \xi_0) + H_2(\xi_0)\chi(t, s, \xi_0) = 0, \quad \chi(s, s, \xi_0) = I, \]

satisfying
\[ |\chi(t, s, \xi_0)| \leq C_1 \exp(-C_2(t-s)) \quad \text{for all} \ 0 \leq s \leq t. \] (2.48)

By Duhamel’s principle and Gronwall’s inequality, it follows from (2.47), using the fact that
\[ |H_2(\xi) - H_2(\xi_0)| < \varepsilon \quad \text{when} \ \xi \ \text{is in a small neighbourhood of} \ \xi_0, \]
that
\[ |Z(t, \xi)| \leq C_1 \exp(-C_2t) \exp(2C_1 \varepsilon t)|V_0(\xi)|. \] (2.49)

Now by choosing a suitable \( \varepsilon \) such that \( 2C_1 \varepsilon < \frac{1}{2}C_2 \), we can obtain the same result as that of (2.45) for all \( \xi \) in the small neighbourhood of \( \xi_0 \). By using the compactness of \( \{ \sigma \leq |\xi| \leq N \} \) we can prove that (2.45) holds for all \( \xi \in \{ \sigma \leq |\xi| \leq N \} \).

By using the backward transformation from \( W \) to \( V \) we find, from (2.39) and (2.45), that
\[ |V(t, \xi)| \leq C_1 \exp(-C_2t)|V_0(\xi)|. \]

This completes the proof of proposition 2.7.

3. Proof of theorems 1.1 and 1.2

In this section, we shall prove theorems 1.1 and 1.2.
3.1. Proof of theorem 1.1

In order to prove theorem 1.1, namely, to derive an $L^p-L^q$ decay estimate for $y^p$, $\theta$ and $q$, we first will derive an $L^2-L^2$ decay estimate and try to derive an $L^1-L^\infty$ decay estimate of $y^p$, $\theta$ and $q$, successively. Then, by the interpolation theorem, we obtain the $L^p-L^q$ decay estimates.

From propositions 2.3, 2.5 and 2.7 we can easily obtain

$$
\|V(t,\xi)\|_{L^2(R^3)} \leq C \|\phi(\xi)V_0(\xi)\|_{L^\infty(R^3)},
$$

(3.1)

where $C$ represents a positive constant.

Subsequently, we will derive the $L^1-L^\infty$ decay estimate for $F^{-1}(V(t,\xi))$.

Letting $\phi(\xi) \in C_0^\infty(R)$ and $\psi(\xi) \in C_0^\infty(R)$ with

$$
\phi(\xi) = \begin{cases} 
1, & |\xi| \leq \sigma, \\
0, & |\xi| > \sigma + 1
\end{cases}
$$

and

$$
\psi(\xi) = \begin{cases} 
1, & |\xi| \geq N + 1, \\
0, & |\xi| < N,
\end{cases}
$$

we then have the following proposition.

**Proposition 3.1.** Let $V(t,\xi) = (V_1, V_2, \ldots, V_{10})^T$ be the solution to the Cauchy problem (2.3) with the initial data $V_0 = (V_{1,0}, V_{2,0}, \ldots, V_{10,0})^T$ and $F^{-1}(V(t,\xi))$ $(i = 1, 2)$, where $V_1(t,\xi) = (V_1, V_2, V_3)^T$ and $V^2(t,\xi) = (V_4, V_5, V_6)^T$, is rotation-free for all $t \geq 0$. Then $V(t,\xi)$ has the following $L^1-L^\infty$ estimates:

$$
\begin{align*}
\|F^{-1}(\phi(\xi)V(t,\xi))\|_{L^\infty(R^3)} & \leq C (1 + t)^{-3/2} \|F^{-1}(\phi(\xi)V_0(\xi))\|_{L^1(R^3)}, \\
\|F^{-1}((1 - \phi(\xi))V(t,\xi))\|_{L^\infty(R^3)} & \leq C_1 \exp(-C_2t) \|F^{-1}(\langle \xi \rangle^2(1 - \phi(\xi))V_0(\xi))\|_{L^1(R^3)},
\end{align*}
$$

(3.2)

where $\langle \xi \rangle^2 := 1 + |\xi|^2$, $C$ and $C_1$ are positive constants.

**Proof.** From proposition 2.3, we obtain, for $|\xi| \leq \sigma$, that

$$
V_k(t,\xi) = \sum_{r=5}^{10} \sum_{l=1}^{10} \exp(-\nu_r(\xi)t)C_{rlk}(\xi)V_{l,0}(\xi), \quad 1 \leq k \leq 10,
$$

(3.3)

where $C_{rlk}(\xi)$ tend to constants $C^0_{rlk}$ as $|\xi| \to 0$ and $C_{rlk}(\xi)$ are bounded on $|\xi| \leq \sigma$. By a direct computation, we have, from (3.3),

$$
\|F^{-1}(\phi(\xi)V_k(t,\xi))\|_{L^\infty(R^3)} \leq C \|\phi(\xi)V_k(t,\xi)\|_{L^1(R^3)} \leq B_1,
$$

(3.4)

where

$$
B_1 = \sum_{r=5}^{10} \sum_{l=1}^{10} \|\exp(-\nu_r(\xi)t)C_{rlk}(\xi)\phi(\xi)V_{l,0}(\xi)\|_{L^1(R^3)}.
$$

(3.5)

Consider lemma 2.2. By a direct computation we have

$$
B_1 \leq C (1 + t)^{-3/2} \sum_{l=1}^{10} \|\phi(\xi)V_{l,0}(\xi)\|_{L^\infty(R^3)} \leq C (1 + t)^{-3/2} \|F^{-1}(\phi(\xi)V_0(\xi))\|_{L^1(R^3)}.
$$

(3.6)
Consequently, we find from (3.4)–(3.6) that
\[ \|F^{-1}(\phi(\xi)V_k(t, \xi))\|_{L^\infty(\mathbb{R}^3)} \leq C(1 + t)^{-3/2}\|F^{-1}(\phi(\xi)V_0(\xi))\|_{L^1(\mathbb{R}^3)}. \] (3.7)
From proposition 2.5, we know that, for \(|\xi| \geq N\),
\[ V_k(t, \xi) = \sum_{r=5}^{10} \sum_{l=1}^{10} \exp(-\nu_r(\xi)t)C_{rlk}(\xi)V_{l,0}(\xi), \] (3.8)
where \(C_{rlk}(\xi)\) tend to constants \(C_{rlk}\) if \(|\xi| \to \infty\) and these functions are bounded for \(|\xi| \geq N\).

Taking account of lemma 2.4 and the equivalence of \(|\xi|\) with \((1 + |\xi|^2)^{1/2}\) when \(|\xi| \geq N\) for a positive \(N\), we obtain the following estimate:
\[ \|F^{-1}(\psi(\xi)V_k(t, \xi))\|_{L^\infty(\mathbb{R}^3)} \leq C\|\psi(\xi)V_k(t, \xi)\|_{L^1(\mathbb{R}^3)} \leq H_1, \] (3.9)
where
\[ H_1 = \sum_{r=5}^{10} \sum_{l=1}^{10} \|\exp(-\nu_r(\xi)t)C_{rlk}(\xi)\psi(\xi)V_{l,0}(\xi)\|_{L^1(\mathbb{R}^3)}. \]
Consider lemma 2.4. We have
\[ H_1 \leq C \exp(-C_2 t)\|F^{-1}(\psi(\xi)\xi^2V_0(\xi))\|_{L^1(\mathbb{R}^3)}. \] (3.10)
Consequently, from (3.9) and (3.10) we obtain
\[ \|F^{-1}(\psi(\xi)V_k(t, \xi))\|_{L^\infty(\mathbb{R}^3)} \leq C \exp(-C_2 t)\|F^{-1}(\psi(\xi)\xi^2V_0(\xi))\|_{L^1(\mathbb{R}^3)}. \] (3.11)
Similarly, we can obtain from proposition 2.7 and (2.28) that, for \(\sigma \leq |\xi| \leq N\),
\[ \|F^{-1((1 - \phi(\xi) - \psi(\xi))V_k(t, \xi))\|_{L^\infty(\mathbb{R}^3)} \leq C \exp(-C_2 t)\|F^{-1((1 - \phi(\xi) - \psi(\xi))\xi^2V_0(\xi))\|_{L^1(\mathbb{R}^3)}. \] (3.12)
Note that \(V(t, \xi) = \phi(\xi)V(t, \xi) + \psi(\xi)V(t, \xi) + (1 - \phi(\xi) - \psi(\xi))V(t, \xi)\). We conclude, by considering (3.7), (3.11) and (3.12), that proposition 3.1 holds.

Now let us prove theorem 1.1. First, from (3.1) we can obtain, by using Parseval’s identity,
\[ \|F^{-1}(\phi(\xi)V(t, \xi))\|_{L^2(\mathbb{R}^3)} \leq C\|F^{-1}(\phi(\xi)V_0(\xi))\|_{L^2(\mathbb{R}^3)}. \] (3.13)
By using the interpolation theorem between (3.7) and (3.13), we have
\[ \|F^{-1}(\phi(\xi)V(t, \xi))\|_{L^q(\mathbb{R}^3)} \leq C(1 + t)^{-(3/2)(1/p - 1/q)}\|F^{-1}(\phi(\xi)V_0(\xi))\|_{W^{N,p}(\mathbb{R}^3)}, \] (3.14)
where \(p^{-1} + q^{-1} = 1, 2 \leq q \leq \infty, N \geq 3(1 - 2q^{-1})\) and \(C\) is a positive constant.
Similarly, we can obtain from (3.11)–(3.13) that
\[ \|F^{-1((1 - \phi(\xi))V(t, \xi))\|_{L^q(\mathbb{R}^3)} \leq C \exp \left(-C_2 \left(\frac{1}{p} - \frac{1}{q}\right)t\right)\|F^{-1((1 - \phi(\xi))\xi^2V_0(\xi))\|_{W^{N,p}(\mathbb{R}^3)}, \] (3.15)
where \(p^{-1} + q^{-1} = 1, 2 \leq q \leq \infty, N \geq 3(1 - 2q^{-1})\) and \(C\) is a positive constant.
From (3.14) and (3.15), we conclude that theorem 1.1 holds.

3.2. Proof of theorem 1.2

By considering the decay estimates (2.1) for \( y^s \) and the decay estimates (3.14) and (3.15) for \( y^p, \theta \) and \( q \), we see that theorem 1.2 holds.

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References


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