L^p - L^q DECAY ESTIMATES FOR THE CAUCHY PROBLEM
OF LINEAR THERMOELASTIC SYSTEMS WITH SECOND SOUND
IN ONE SPACE VARIABLE

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Abstract. L^p - L^q decay estimates of solutions to the Cauchy problem of linear thermoelastic systems with second sound in one space variable will be studied in this paper. First, by dividing the frequency of phase space of the Fourier transformation into different regions, the asymptotic behavior of characteristic roots of the coefficient matrix is obtained by carefully analyzing the effect of the different regions. Second, with the help of the information on the characteristic roots and by using the interpolation theorem, the L^p - L^q decay estimate of solutions to the Cauchy problem of the linear thermoelastic system with second sound in one space variable is obtained.

1. Introduction. The thermoelastic system describes the elastic and thermal behavior of elastic heat conductive media, in particular the reciprocal action between elastic stresses and temperature differences. They are hyperbolic-parabolic coupled systems (1–4, 6, 9, 14). Although the classical thermoelasticity theory, based on the Fourier law of heat conduction, predicts that a thermal disturbance of material conducting heat will be propagated with infinite speeds, experiments on certain dielectric crystals at very low temperature have produced results that show the thermal disturbance is transmitted as “wave-like” pulses with finite speeds (see 2, 3, and references therein). This phenomenon is known as second sound. The effect of the second sound makes the thermoelastic system be a purely hyperbolic, but damped one (cf. 5, 7, 10, 11, 14, and references therein) based on the Cattaneo’s law for heat conduction instead of Fourier’s law. Thus the paradox of infinite propagation speeds of heat pulse is overcome, which is very important in some applications such as pulsed laser heating of solids (cf. 8, 15).
are some interesting works devoted to the thermoelastic system with second sound (cf. [5], [7], [10], [11], [14], and references therein). For example, in [5] Gurtin and Pipkin developed a general theory of heat flow in rigid bodies with memory which predicts finite propagation speeds for thermal disturbance. Tarabek in [14] obtained the existence of smooth solutions of linear thermoelastic systems with second sound in both the case of an unbounded body and the case of a bounded body with pinned and insulated ends in one space variable by using the energy method. Racke in [11] studied the asymptotic behavior of solutions of linear thermoelastic systems with second sound in two and three space variables. He proved an exponential stability for the purely, but lightly damped, hyperbolic system with Dirichlet type boundary conditions. In [10] Racke proved that the dissipativity given through heat conduction is strong enough to exponentially stabilize all components \((u, \theta, q)\) in linear and nonlinear systems in one space variable.

The \(L^p - L^q\) decay estimate can qualitatively describe the long time behavior of solutions. It is well known that to establish the \(L^p - L^q\) decay estimate is the crucial step in studying the existence of global solution with small initial data for nonlinear problems (cf. [6], [9], and references therein). There is a rich literature concerned with \(L^p - L^q\) decay estimates for the thermoelastic system of the hyperbolic-parabolic coupled type (cf. [6], [9], [13], [17], [19], and references therein). Recently, Racke, Reissig, and the authors found a new method by frequency analysis in phase space to decouple the hyperbolic-parabolic coupled system to obtain the \(L^p - L^q\) decay estimate, and to study the propagation of singularity of solutions of linear and nonlinear problems (cf. [4], [12]–[13], [16]–[18], and references therein). For instance, in [13] Reissig and Wang investigated the well-posedness, propagation of singularities, and \(L^p - L^q\) decay estimates of solutions to the Cauchy problem of linear thermoelastic systems of type III in 1-D by frequency analysis. By dividing the phase space into three distinct regions, they also (in [17]) studied the long time behavior of solutions of the Cauchy problem for the linear thermoelastic system with time-dependent coefficients in one space variable and obtained a parabolic-type \(L^p - L^q\) decay estimate.

The purpose of this paper is to study the \(L^p - L^q\) decay estimate of solutions of the Cauchy problem for a linear thermoelastic system with second sound in one space variable by frequency analysis. First, the phase space will be divided into small, middle, and large regions. By carefully analyzing the effect of different regions, an efficient method will be found to diagonalize the principal part of the system correspondingly. Second, by studying the weakly decoupled system, the \(L^p - L^q\) decay estimate of solutions of the Cauchy problem for the linear thermoelastic system with second sound in one space variable will be obtained.

Let us consider the following Cauchy problem for a linear thermoelastic system with second sound in one space variable:

\[
\begin{align*}
\frac{u_{tt}}{\alpha^2} + \beta \theta_x &= 0, \\
\theta_t + \gamma q_x + \delta u_{xt} &= 0, \\
\tau q_t + q + \kappa \theta_x &= 0, \\
u(0, x) &= u_0(x), \quad u_t(0, x) = u_1(x), \\
\theta(0, x) &= \theta_0(x), \quad q(0, x) = q_0(x),
\end{align*}
\] (1.1)
where \( u \) and \( \theta \) are the displacement and the temperature difference of the elastic media, \( q \) is the heat flux, \( \alpha, \beta, \gamma, \delta, \tau, \) and \( \kappa \) are all positive constants, and the following notations

\[
\begin{align*}
\{ u_t = \frac{\partial u}{\partial t}, & \quad u_{xx} = \frac{\partial^2 u}{\partial x^2}, \quad u_{tx} = \frac{\partial^2 u}{\partial x \partial t}, \\
\theta_x = \frac{\partial \theta}{\partial x}, & \quad \theta_t = \frac{\partial \theta}{\partial t}, \quad q_x = \frac{\partial q}{\partial x}, \quad q_t = \frac{\partial q}{\partial t}
\end{align*}
\]

are used here and will also be used in the remainder of this paper.

Letting

\[
\begin{align*}
\hat{u}^+ &= \hat{u}_t + i \alpha \xi \hat{u}, \\
\hat{u}^- &= \hat{u}_t - i \alpha \xi \hat{u},
\end{align*}
\]

where \( \hat{u} \) is the Fourier transformation of \( u \) with respect to \( x \), we know from (1.1) that \( V = (\hat{u}^+, \hat{u}^-, \theta, q)^T \) satisfies the following Cauchy problem of a first order system:

\[
\begin{align*}
\begin{cases}
\partial_t V + i \xi A_1 V + A_0 V = 0, \\
V(0, x) := V_0(x) = (\hat{u}_1 + i \alpha \xi \hat{u}_0, \hat{u}_1 - i \alpha \xi \hat{u}_0, \hat{\theta}_0, \hat{q}_0)^T,
\end{cases}
\end{align*}
\]

where

\[
A_1 = \begin{pmatrix}
-\alpha & 0 & \beta & 0 \\
0 & \alpha & \beta & 0 \\
\frac{i}{2} & \frac{i}{2} & 0 & \gamma \\
0 & 0 & \frac{i}{2} & 0
\end{pmatrix}, \quad A_0 = \text{diag}\{0, 0, 0, \frac{1}{\tau}\}.
\]

The main results of this paper are as follows:

**Theorem 1.1.** Denote by the matrix \( B := i \xi A_1 + A_0 \) and \( \nu_k (1 \leq k \leq 4) \), its eigenvalues. Let \( \chi = \chi(t, s, \xi), \) \( 0 \leq s \leq t, \) be the fundamental matrix to the operator \( \partial_t + B \) with \( \chi(s, s, \xi) = I \). Then there are constants \( \sigma \leq 1, N \gg 1, \) constant matrices \( K_j (1 \leq i \leq 4), L_j \) \( (j = 1, 2), \) and invertible matrices \( M_i \) \( (i = 1, 2) \) such that \( \chi \) has the following properties:

\[
\begin{align*}
\chi(t, s, \xi) &= Q_1^{-1} \text{diag}(\exp(-\nu_1(\xi)(t-s)), ..., \exp(-\nu_4(\xi)(t-s)))Q_1 \quad \text{as } |\xi| \leq \sigma \ll 1, \\
\chi(t, s, \xi) &= Q_2^{-1} \text{diag}(\exp(-\nu_1(\xi)(t-s)), ..., \exp(-\nu_4(\xi)(t-s)))Q_2 \quad \text{as } |\xi| \geq N \gg 1, \\
|\chi(t, s, \xi)| &\leq C_1 \exp(-C_2(t-s)) \quad \text{as } \sigma \leq |\xi| \leq N,
\end{align*}
\]

where \( C_1, C_2 \) are two positive constants being independent of \( (t, s, \xi) \) and

\[
\begin{align*}
Q_1 &= M_1(I + K_2 \xi)(I + K_3 \xi^2)K_2(I + K_1 \xi), \\
Q_2 &= M_2(I + L_2 \xi^{-1})L_1.
\end{align*}
\]

**Theorem 1.2.** Let \( (u, \theta, q) \) be the unique solution to the Cauchy problem (1.1). Then it satisfies the following \( L^p - L^q \) decay estimate

\[
\| (u_t, u_{xx}, \theta, q) \|_{L^q(R)} \leq C(1 + t)^{-\frac{\gamma}{2} + \frac{1}{p} - \frac{1}{q}}\| (u_1, u_0, \theta_0, q_0) \|_{W^{N,p}(R)},
\]

where \( N \geq (1 - \frac{2}{q}) \), \( \frac{1}{p} + \frac{1}{q} = 1, 2 \leq q \leq \infty, \) \( C \) is a positive constant, and \( W^{N,p}(R) \) is the classical Sobolev space based on \( L^p(R) \).

The remainder of this paper is arranged as follows: In \( \S 2 \), we shall diagonalize the principal terms of system (1.2) to obtain the asymptotic behavior of characteristic roots of \( B := i \xi A_1 + A_0 \) for the higher and lower frequencies respectively. Meanwhile, we can also obtain some information on the left and right eigenvectors of the matrix \( B \). Subsequently, by the aid of the information, Theorems 1.1 and 1.2 will be proved in \( \S 3 \).
2. Diagonalizing the principal terms in the system (1.2). In this section, we shall carefully analyze the effect of frequency in different regions of phase space to obtain the asymptotic behavior of the eigenvalues of the matrix $B$ to diagonalize the matrix $B$.

2.1. In the region $|\xi| \leq \sigma \ll 1$. We restrict our discussion in this subsection to the region $|\xi| \leq \sigma \ll 1$.

The matrix $A_0$ is diagonal already; let us diagonalize $A_1$ in the following procedure.

Let $V^1 = (I + K_1 \xi)V$, where the constant matrix $K_1$ will be determined later. Then we have

$$
\partial_t V^1 = (I + K_1 \xi)(-i \xi A_1 V - A_0 V)
= -A_0 V^1 + (\xi [A_0, K_1] - i \xi A_1) V^1 - A_1^2(\xi) V^1,
$$

(2.1)

where $[,]$ means the commutator of two related matrices and $A_1^2(\xi) = (C_{ij})_{4 \times 4}$ with $C_{ij} = O(|\xi|^2)$ at least when $\xi \to 0$.

Choosing

$$
K_1 = i \begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & -\tau \gamma \\
0 & 0 & \kappa & 0
\end{pmatrix}
$$

and letting $A_1^1 := \xi [K_1, A_0] + i \xi A_1$, by a direct computation we have

$$
A_1^1 = \begin{pmatrix}
-\alpha & 0 & \beta & 0 \\
0 & \alpha & \beta & 0 \\
\delta & \delta & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}.
$$

Thus, we know that $V^1$ satisfies

$$
\partial_t V^1 + A_0 V^1 + A_1^1 V^1 + A_1^2(\xi) V^1 = 0.
$$

(2.2)

Subsequently, we shall diagonalize $A_1^1$ in the following procedure.

Letting

$$
D = \begin{pmatrix}
-\alpha & 0 & \beta \\
0 & \alpha & \beta \\
\delta & \delta & 0
\end{pmatrix},
$$

by a direct computation we know

$$
|\lambda I - D| = \lambda^3 - (\beta \delta + \alpha^2) \lambda.
$$

Obviously, there are three distinct eigenvalues of the matrix $D$ by the aid of assumptions in (1.1),

$$
\lambda_1 = 0, \quad \lambda_{2/3} = \pm \sqrt{\beta \delta + \alpha^2} =: \pm b.
$$

Let $l_k$, $r_k$ denote the left and right eigenvectors of $D$ with respect to $\lambda_k$ respectively. We can easily obtain

$$
\begin{cases}
l_1 = C_{l1}(\delta, -\delta, 2\alpha), & r_1 = C_{r1}(\beta, -\beta, \alpha)^T, \\
l_2 = C_{l2}(\frac{\delta}{b + \alpha}, \frac{\delta}{b - \alpha}, 2), & r_2 = C_{r2}(\frac{\beta}{b + \alpha}, \frac{\beta}{b - \alpha}, 1)^T, \\
l_3 = C_{l3}(\frac{i}{\alpha - \beta}, -\frac{i}{\alpha + \beta}, 2), & r_3 = C_{r3}(\frac{\beta}{\alpha - \beta}, -\frac{\beta}{\alpha + \beta}, 1)^T,
\end{cases}
$$
where \( C_{jk} \) \((1 \leq j, k \leq 3)\), are constants satisfying
\[
l_{j}r_{k} = \delta_{jk},
\]
which implies \( C_{i}C_{ri} > 0 \) \((1 \leq i \leq 3)\).

Letting
\[
\begin{align*}
K_{2} &= \begin{pmatrix}
\ell_{1} & 0_{3 \times 1} \\
\ell_{2} & 0_{3 \times 1} \\
\ell_{3} & 1
\end{pmatrix},
\end{align*}
\]
then \( V^{2} = K_{2}V^{1} \) satisfies
\[
\partial_{t}V^{2} + A_{0}^{2}V^{2} + i\xi A_{1}^{2}V^{2} + A_{2}^{2}(\xi)V^{2} = 0,
\]
where
\[
A_{0}^{2} = A_{0}, \quad A_{1}^{2} = \text{diag}\{0, b, -b, 0\}
\]
and
\[
A_{2}^{2}(\xi) = (C_{ij})_{4 \times 4} \text{ with } C_{ij} = O(|\xi|^{2}) \text{ at least when } \xi \to 0.
\]

Let \( B = A_{0} + i\xi A_{1} \), then we have the following lemma:

**Lemma 2.1.** The characteristic roots \( \nu_{k} = \nu_{k}(\xi) \) \((1 \leq k \leq 4)\), of the matrix \( B \) behave for \( |\xi| \leq \sigma \ll 1 \) as
\[
\begin{align*}
\nu_{1} &= 2\gamma\kappa\xi^{2}C_{11}C_{r1} + O(|\xi|^{3}), \\
\nu_{2} &= i\xi b + 2\gamma\kappa\xi^{2}C_{12}C_{r2} + O(|\xi|^{3}), \\
\nu_{3} &= -i\xi b + 2\gamma\kappa\xi^{2}C_{13}C_{r3} + O(|\xi|^{3}), \\
\nu_{4} &= \frac{1}{\tau} + (\frac{1}{\tau} - 2\kappa\gamma)\xi^{2} + O(|\xi|^{3}).
\end{align*}
\]

**Proof.** By a direct computation for \( A_{1}^{2}(\xi) \) in (2.1) we know that
\[
A_{2}^{2}(\xi) = i\xi^{2}[K_{1}, A_{1}] - \xi^{2}K_{1}A_{0}K_{1}(I + K_{1})^{-1} - i\xi^{3}K_{1}A_{1}K_{1}(I + K_{1})^{-1}
\]
\[
= \xi^{2}\begin{pmatrix}
0 & 0 & 0 & -\gamma/\beta \\
0 & 0 & 0 & -\gamma/\beta \\
0 & 0 & \gamma/\kappa & 0 \\
-\kappa & -\kappa & 0 & -2\kappa\gamma
\end{pmatrix} + O(|\xi|^{3}).
\]

From the derivation of \( A_{2}^{2}(\xi) \) in (2.3), by a direct computation we obtain
\[
A_{2}^{2}(\xi) = \xi^{2}\begin{pmatrix}
2\alpha^{2}\gamma\kappa C_{11}C_{r1} & 2\alpha\gamma\kappa C_{12}C_{r1} & 2\alpha\gamma\kappa C_{13}C_{r1} & 0 \\
2\alpha\gamma\kappa C_{12}C_{r1} & 2\gamma\kappa C_{12}C_{r2} & 2\gamma\kappa C_{13}C_{r2} & -2b\gamma\tau C_{12} \\
2\alpha\gamma\kappa C_{13}C_{r1} & 2\gamma\kappa C_{13}C_{r2} & 2\gamma\kappa C_{13}C_{r3} & 2b\gamma\tau C_{13} \\
0 & -b\kappa C_{r2} & b\kappa C_{r3} & -2\kappa\gamma
\end{pmatrix} + O(|\xi|^{3})
\]
\[
=: \xi^{2}\tilde{A}_{2}^{2} + \tilde{A}_{3}^{2}(\xi),
\]
where \( \tilde{A}_{2}^{2}(\xi) = K_{2}A_{2}^{2}K_{2}^{-1} \), \( \tilde{A}_{3}^{2}(\xi) = (C_{ij})_{4 \times 4} \) with \( C_{ij} = O(|\xi|^{3}) \) when \( \xi \to 0 \).

In the following procedure we shall successively diagonalize \( \tilde{A}_{2}^{2}(\xi) \) with the help of \( A_{0}^{2} \) and \( A_{1}^{2} \).
Let \( V^3 = (I + K_3 \xi^2) V^2 \) where the constant matrix \( K_3 \) will be determined later. Then from (2.3) we know that \( V^3 \) satisfies

\[
\partial_t V^3 = (I + K_3 \xi^2)(-A_0^3 - i\xi A_1^3 - \xi^2 A_2^3 - A_3^3(\xi))V^2
\]

\[
= -A_0^3 V^3 - i\xi A_1^3 V^3 - (\xi^2 K_3, A_0^3) + \xi^2 A_2^3 V^3 - A_3^3(\xi) V^3,
\]

where \( A_3^3(\xi) = (C_{ij})_{4 \times 4} \) with \( C_{ij} = O(|\xi|^3) \) at least when \( \xi \to 0 \).

From (2.5) we have

\[
\partial_t V^3 + A_0^3 V^3 + i\xi A_1^3 V^3 + \xi^2 A_2^3 V^3 + A_3^3(\xi) V^3 = 0
\]  

where \( A_0^3 = A_0^2, A_1^3 = A_1^2, A_2^3 = [K_3, A_0^3] + A_2^2 \).

By choosing

\[
K_3 = \begin{pmatrix}
0 & 0 & 0 & 2b\gamma\tau^2 C_{12} \\
0 & 0 & 0 & -2b\gamma\tau^2 C_{13} \\
0 & -b\kappa\tau C_{12} & b\kappa\tau C_{13} & 0 \\
0 & 0 & 0 & -2\kappa^2 \\
\end{pmatrix},
\]

we have, by a direct computation, that

\[
A_2^3 = \begin{pmatrix}
2\alpha^2\gamma\kappa C_{11} C_{12} & 2\alpha\gamma\kappa C_{11} C_{12} & 2\alpha\gamma\kappa C_{11} C_{13} & 0 \\
2\alpha\gamma\kappa C_{12} C_{12} & 2\gamma\kappa C_{12} C_{12} & 2\gamma\kappa C_{12} C_{13} & 0 \\
2\alpha\gamma\kappa C_{13} C_{12} & 2\gamma\kappa C_{13} C_{12} & 2\gamma\kappa C_{13} C_{13} & 0 \\
0 & 0 & 0 & -2\kappa\gamma \\
\end{pmatrix},
\]

where \( A_3^3 = [K_3, A_0^3] + A_2^2 \).

Subsequently, we shall diagonalize \( A_2^3 \) by the aid of \( A_1^3 \).

Let

\[
K_4 = i \begin{pmatrix}
0 & k_{12} & k_{13} & 0 \\
k_{21} & 0 & k_{23} & 0 \\
k_{31} & k_{32} & 0 & 0 \\
0 & 0 & 0 & 1 \\
\end{pmatrix},
\]

where constants \( k_{ij} (1 \leq i, j \leq 4) \) will be determined later. From (2.6), we know that \( V^4 = (I + K_4 \xi) V^3 \) satisfies

\[
\partial_t V^4 = (I + K_4 \xi)(-A_0^3 - i\xi A_1^3 - \xi^2 A_2^3 - A_3^3(\xi))V^3
\]

\[
= -A_0^3 V^4 - i\xi A_1^3 V^4 - (i\xi A_1^3 + \xi[K_4, A_0^3])V^4
\]

\[
- (i\xi^2 K_4, A_3^3) - \xi^2 K_4 A_0^3 K_4 + \xi^2 A_2^3 V^4 - A_3^3(\xi) V^4,
\]

where \( A_3^3(\xi) = (C_{ij})_{4 \times 4} \) with \( C_{ij} = O(|\xi|^3) \) at least when \( \xi \to 0 \).

By a direct computation, we obtain

\[
[K_4, A_0^3] = 0_{4 \times 4},
\]

\[
i[K_4, A_1^3] - K_4 A_0^3 K_4 = \begin{pmatrix}
0 & -k_{12}b & k_{13}b & 0 \\
k_{21}b & 0 & 2k_{23}b & 0 \\
-k_{31}b & -2k_{32}b & 0 & 0 \\
0 & 0 & 0 & \frac{1}{7}
\end{pmatrix}.
\]
Letting
\[
\begin{align*}
k_{12} &= \frac{2\alpha \kappa \gamma C_1 C_{r1}}{\beta}, & k_{13} &= \frac{-2\alpha \kappa \gamma C_1 C_{r2}}{\beta}, & k_{21} &= \frac{-2\alpha \kappa \gamma C_1 C_{r3}}{\beta}, \\k_{23} &= \frac{-\kappa \gamma C_2 C_{r2}}{\beta}, & k_{31} &= \frac{2\alpha \kappa \gamma C_2 C_{r1}}{\beta}, & k_{32} &= \frac{\kappa \gamma C_2 C_{r3}}{\beta},
\end{align*}
\]
we obtain from (2.7) and (2.9) that
\[
A_4 = \text{diag}\{2\alpha^2 \gamma \kappa C_1 C_{r1}, 2\gamma \kappa C_2 C_{r2}, 2\gamma \kappa C_3 C_{r3}, \frac{1}{T} - 2\gamma \kappa\},
\]
where \(A_4 := A_2^3 + i[K_4, A_3^1] - K_4A_3^3K_4.\)

Thus, from (2.8) and (2.10) we know that \(V^4\) satisfies
\[
\partial_t V^4 + A_0^4 V^4 + i \xi A_1^4 V^4 + \xi^2 A_2^4 V^4 + A_3^4(\xi) V^4 = 0,
\]
where \(A_0^4 = A_0^3, A_1^4 = A_1^3.\)

By the similarity of matrix, we know from (2.11) that Lemma 2.1 holds. \(\square\)

**Corollary 2.1.** The matrices \(M_1\) and \(R_1\) composed by the left and right eigenvectors to the characteristic roots of \(B^4 := A_4^4 + i \xi A_1^4 + \xi^2 A_2^4 + A_3^4(\xi)\) can be written for \(|\xi| \leq \sigma \ll 1\) in the form \(M_1 = (l_{jk}(\xi))_{j,k=1}^{23}\) and \(R_1 = (r_{jk}(\xi))_{j,k=1}^{23}\) with
\[
\begin{align*}
l_{jk}(\xi) &= O(|\xi|^3), & r_{jk}(\xi) &= O(|\xi|^3) \text{ for } |\xi| \to 0 \text{ and } j \neq k.
\end{align*}
\]

**Proof.** From (2.11), we have for \(\nu_1\) that
\[
(\nu_1 I - A_0^4 - i \xi A_1^4 - \xi^2 A_2^4 - A_3^4(\xi)) R_1 = 0,
\]
which implies that \(R_1 = (r_{11}, r_{21}, r_{31}, r_{41})^T\) satisfies
\[
\begin{pmatrix}
Z_2 & -d_{23}(\xi) & -d_{24}(\xi) \\
-d_{32}(\xi) & Z_3 & -d_{34}(\xi) \\
-d_{42}(\xi) & -d_{43}(\xi) & Z_4
\end{pmatrix}
\begin{pmatrix}
r_{12} \\
r_{22} \\
r_{32}
\end{pmatrix}
= \begin{pmatrix}
d_{21}(\xi) \\
d_{31}(\xi) \\
-d_{41}(\xi)
\end{pmatrix}
\begin{pmatrix}
r_{11} \\
r_{21} \\
r_{31}
\end{pmatrix}
\]
where
\[
\begin{align*}
Z_2 &= -i \xi b + 2\gamma \kappa \xi^2 (\alpha^2 C_1 C_{r1} - C_{r2}) + O(|\xi|^3), \\
Z_3 &= i \xi b + 2\gamma \kappa \xi^2 (\alpha^2 C_1 C_{r1} - C_{r2}) + O(|\xi|^3), \\
Z_4 &= 2\gamma \kappa \xi \xi^2 C_1 C_{r1} - \frac{1}{\xi} (1 + \xi^2) + 2\gamma \kappa \xi^2 + O(|\xi|^3), \\
d_{ij}(\xi) &= O(|\xi|^3), \quad (2 \leq i,j \leq 4), \quad \text{as } |\xi| \to 0.
\end{align*}
\]

Letting \(r_{11} = 1\), we obtain from the above equation by considering \(b \neq 0\) from (2.3) that
\[
R_1 = (1, r_{21}, r_{31}, r_{41})^T \text{ with } r_{11} = O(|\xi|^2) \text{ as } |\xi| \to 0.
\]

Similarly, we can prove the statements for the other right and left eigenvectors. \(\square\)

**Proposition 2.1.** The solution to the Cauchy problem
\[
\begin{align*}
\partial_t V + i \xi A_1 V + A_0 V &= 0, \\
V(0, \xi) &= V_0(\xi),
\end{align*}
\]
has the following representation in \(|\xi| \leq \sigma \ll 1:\)
\[
V(t, \xi) = Q_1^{-1} \text{diag}\{\exp(-\nu_1 t), \exp(-\nu_2 t), \exp(-\nu_3 t), \exp(-\nu_4 t)\} Q_1 V_0(\xi),
\]
(2.12)
where \( Q_1 = M_1(I + K_4 \xi)(I + K_3 \xi^2)K_2(I + K_1 \xi) \), the matrices \( M_1, K_i \) and \( \nu_j \) \((1 \leq j \leq 4)\), are given before.

**Proof.** From the representation of \( B \) and the previous procedure of diagonalizing \( B \), we know from
\[
\partial_t V^4 + B^4 V^4 = 0, \tag{2.13}
\]
with \( V_0^4 = (I + K_4 \xi)(I + K_3 \xi^2)K_2(I + K_1 \xi)V_0 \) and \( B^4 = A_0^4 + i \xi A_1^4 + \xi^2 A_2^4 + A_3^4(\xi) \), that \( W = M_1 V^4 \) satisfies
\[
\begin{align*}
\partial_t W + DW &= 0, \\
W_0(\xi) &= M_1(I + K_4 \xi)(I + K_3 \xi^2)K_2(I + K_1 \xi)V_0,
\end{align*}
\]
where \( D = \text{diag}\{\nu_1, \nu_2, \nu_3, \nu_4\} \).

By a direct computation, we have
\[
W = \text{diag}\{\exp(-\nu_1(\xi)t), \exp(-\nu_2(\xi)t), \exp(-\nu_3(\xi)t), \exp(-\nu_4(\xi)t)\}W_0. \tag{2.14}
\]

Using the backward transformation from \( W \) to \( V \), from (2.14) we can obtain the representation (2.12).

2.2. In the region \( |\xi| \geq N \gg 1 \). In this subsection, we will restrict our discussion to the region \( |\xi| \geq N \gg 1 \). First, we shall diagonalize the main part \( i \xi A_1 \) of (1.2) in this region.

From the representation of \( A_1 \) in (1.2), we have
\[
|\lambda I - A_1| = \lambda^4 - (\alpha^2 + \frac{\gamma \kappa}{\tau} + \beta \delta) \lambda^2 + \frac{\alpha^2 \gamma \kappa}{\tau}. \tag{2.15}
\]

By the aid of the assumptions on the coefficients in (1.1) and a direct computation, we know from (2.15) that the matrix \( A_1 \) has four distinct real characteristic roots:
\[
\lambda_{1/2} = \pm \sqrt{\frac{n + m}{2}}, \quad \lambda_{3/4} = \pm \sqrt{\frac{n - m}{2}},
\]
where
\[
m = \alpha^2 + \frac{\gamma \kappa}{\tau} + \beta \delta, \quad n = \sqrt{\left(\alpha^2 + \frac{\gamma \kappa}{\tau} + \beta \delta\right)^2 - \frac{4 \gamma \kappa \alpha^2}{\tau}}.
\]

Letting \( l_k, r_k \) denote the left and right eigenvectors with respect to \( \lambda_k \) \((1 \leq k \leq 4)\), by a direct computation we have
\[
l_j = C_{l_j}\left(\frac{\delta}{\lambda_j + \alpha}, \frac{\delta}{\lambda_j - \alpha}, 2, \frac{2 \gamma}{\lambda_j}\right), \quad r_i = C_{r_i}\left(\frac{\beta}{\lambda_i + \alpha}, \frac{\beta}{\lambda_i - \alpha}, 1, \frac{\kappa}{\lambda_i \tau}\right),
\]
where \( C_{l_j} \) and \( C_{r_i} \) satisfy
\[
l_j r_i = \delta_{ij} \Leftrightarrow C_{l_j} C_{r_j} = \left(\frac{\beta \delta}{(\lambda_j + \alpha)^2} + \frac{\beta \delta}{(\lambda_j - \alpha)^2} + 2 + \frac{2 \gamma \kappa}{\tau \lambda_j^2}\right) - 1 > 0. \tag{2.16}
\]

Letting \( L_1 = (l_1, l_2, l_3, l_4)^T \), then from (1.2) we know that \( V^1 = L_1 V \) satisfies
\[
\partial_t V^1 + i \xi \Delta V^1 + A_0^1 V^1 = 0, \tag{2.17}
\]
where
\[
\Lambda = \text{diag}\{\lambda_1, \lambda_2, \lambda_3, \lambda_4\} \quad \text{and} \quad A_0^1 = L_1 A_0 L_1^{-1} := (a_{ij})_{4 \times 4} \tag{2.18}
\]
with \( a_{ij} = \frac{2 \gamma \kappa C_{l_i} C_{r_j}}{\tau^2 \lambda_i \lambda_j} \) \((1 \leq i, j \leq 4)\).
Now we shall successively diagonalize $A^0_1$ in the following procedure.

Let $V^2 = (I + L_2 \xi^{-1})V^1$ where the constant matrix $L_2$ will be determined later. Then from (2.17) we have

$$\partial_t V^2 = (I + L_2 \xi^{-1})(-i\xi A - A^0_1)(I - L_2 \xi^{-1} + \xi^{-2} L_2^2 (I + L_2 \xi^{-1})^{-1}) V^2$$

$$= -i\xi \Lambda V^2 - (A^0_1 + i[L_2, \Lambda])V^2 - A^2_{-1}(\xi)V^2,$$

where $A^2_{-1}(\xi) = (C_{ij})_{4 \times 4}$ with $C_{ij} = O(|\xi|)$ when $|\xi| \to \infty$.

Setting

$$L_2 = i \begin{pmatrix} 0 & l_{12} & l_{13} & l_{14} \\
 l_{21} & 0 & l_{23} & l_{24} \\
 l_{31} & l_{32} & 0 & l_{34} \\
 l_{41} & l_{42} & l_{43} & 0 \end{pmatrix},$$

we can obtain by a direct computation that

$$A^0_2 := A^0_1 + i[L_2, \Lambda]$$

$$= \begin{pmatrix}
a_{11} & a_{12} - (\lambda_2 - \lambda_1) l_{12} & a_{13} - (\lambda_3 - \lambda_1) l_{13} & a_{14} - (\lambda_4 - \lambda_1) l_{14} \\
 a_{21} - (\lambda_1 - \lambda_2) l_{21} & a_{22} & a_{23} - (\lambda_3 - \lambda_2) l_{23} & a_{24} - (\lambda_4 - \lambda_2) l_{24} \\
 a_{31} - (\lambda_1 - \lambda_3) l_{31} & a_{32} - (\lambda_2 - \lambda_3) l_{32} & a_{33} & a_{34} - (\lambda_4 - \lambda_3) l_{34} \\
 a_{41} - (\lambda_1 - \lambda_4) l_{41} & a_{42} - (\lambda_2 - \lambda_4) l_{42} & a_{43} - (\lambda_3 - \lambda_4) l_{43} & a_{44} \\
\end{pmatrix}. \quad (2.20)$$

In terms of the distinction of the characteristic roots $\lambda_i$ ($1 \leq i \leq 4$), of the matrix $A_1$ in (1.2), we can choose $l_{ij}$ appropriately; for example, we may let $l_{ij} = \frac{a_{ij}}{\lambda_j - \lambda_i}$ ($1 \leq i, j \leq 4$ and $i \neq j$) such that

$$A^0_2 = \text{diag}\{a_{11}, a_{22}, a_{33}, a_{44}\}, \quad (2.21)$$

where from (2.16) we know $a_{ii} > 0$ ($1 \leq i \leq 4$).

Thus, from (2.18), (2.19) and (2.21) we know that $V^2$ satisfies

$$\partial_t V^2 + i\xi \Lambda V^2 + A^0_2 V^2 + A^2_{-1}(\xi)V^2 = 0. \quad (2.22)$$

Obviously from the above diagonalizing procedure, we can obtain the following lemma.

**Lemma 2.2.** The characteristic roots $\nu_k$ ($1 \leq k \leq 4$), of the matrix $B = i\xi A_1 + A_0$ behave for $|\xi| \geq N \gg 1$ as

$$\nu_k = \nu_k(\xi) = a_{kk} + i\xi \lambda_k + O(|\xi|^{-1}),$$

where $a_{kk}$ are from $A^0_2$ and $\lambda_k$ are the characteristic roots of $A_1$ in (1.2).

**Corollary 2.2.** The matrices $M_2$ and $R_2$ composed by the left and right eigenvectors to the characteristic roots of $B^2 := i\xi A + A^2_{-1}(\xi)$ can be written for $|\xi| \geq N \gg 1$ in the form $M_2 = (l^2_{jk}(\xi))_{j,k=1}^4$ and $R_2 = (r^2_{jk}(\xi))_{j,k=1}^4$ with

$$\left\{ \begin{array}{l}
l^2_{kk} = r^2_{kk} = 1, \\
r^2_{jk}(\xi) = O(|\xi|^{-1}), \quad l^2_{jk}(\xi) = O(|\xi|^{-1}) \quad \text{for} \quad |\xi| \to \infty \quad \text{and} \quad j \neq k. \end{array} \right.$$ 

The procedure of proving Corollary 2.2 is similar to that of Corollary 2.1. We omit it for simplicity.
Proposition 2.2. The solution to the Cauchy problem (1.2) has the following representation as $|\xi| \geq N \gg 1$,

$$V(t, \xi) = Q_2^{-1} \text{diag}\{\exp(-\nu_1 t), \exp(-\nu_2 t), \exp(-\nu_3 t), \exp(-\nu_4 t)\}Q_2 V_0(\xi),$$  \hspace{1cm} (2.24)

where $Q_2 = M_2(I + L_2 \xi^{-1})L_1$, $L_i \ (i = 1, 2)$ are determined before and $M_2$ is from Corollary 2.2.

Proof. Using Lemma 2.2 and Corollary 2.2, we can prove this proposition in a similar way to that of proving Proposition 2.1. \hfill \Box

2.3. In the region $\sigma \leq |\xi| \leq N$. In this subsection, we will restrict our discussion to this region $\sigma \leq |\xi| \leq N$.

Letting $\nu_k \ (1 \leq k \leq 4)$, denote the characteristic roots of $B = i\xi A_1 + A_0$ in (1.2), from Lemmas 2.1, 2.2, and the following Lemma 2.3 we know with the help of compactness of $\{\sigma \leq |\xi| \leq N\}$ that

$$\text{Re} \nu_k(\xi) \geq C > 0 \ (1 \leq k \leq 4), \text{ for all } \xi \in \{\sigma \leq |\xi| \leq N\},$$  \hspace{1cm} (2.25)

where $C$ represents a constant.

Lemma 2.3. The matrix $B = i\xi A_1 + A_0$ in (1.2) has no purely imaginary eigenvalue $\nu = ia, \ a \neq 0$ in $\{\sigma \leq |\xi| \leq N\}$.

Proof. Let us suppose that $\nu = ia, \ a \neq 0, \ a \in R$ is one of the eigenvalues of $B$ in $\{\sigma \leq |\xi| \leq N\}$. Then we know that $i\alpha$ satisfies

$$|iaI - B| = \begin{vmatrix}
i(a + \alpha \xi) & 0 & -i\beta \xi & 0 \\
0 & -\frac{i\delta \xi}{2} & -i\delta \xi & 0 \\
\frac{i\delta \xi}{2} & i\alpha \xi & i\alpha \xi & -i\gamma \xi \\
0 & 0 & -1 & ia - \frac{1}{\tau}\end{vmatrix} = 0.$$  \hspace{1cm} (2.26)

By a direct computation we obtain from (2.26) that

$$a^2((a - \alpha \xi)(a + \alpha \xi) - \delta \beta \xi^2) - \frac{\gamma \kappa \xi^2}{\tau}((a - \alpha \xi)(a + \alpha \xi))$$

$$+ \frac{ia}{\tau}((a - \alpha \xi)(a + \alpha \xi) - \delta \beta \xi^2) = 0.$$  \hspace{1cm} (2.27)

We deduce from (2.27) that

$$\begin{cases}
\frac{i}{\tau}((a - \alpha \xi)(a + \alpha \xi) - \delta \beta \xi^2) = 0, \\
a^2((a - \alpha \xi)(a + \alpha \xi) - \delta \beta \xi^2) - \frac{\gamma \kappa \xi^2}{\tau}((a - \alpha \xi)(a + \alpha \xi)) = 0.
\end{cases}$$  \hspace{1cm} (2.28)

Due to $a \neq 0$, by substituting the first equation in (2.28) into the second one, we can obtain

$$\frac{\gamma \kappa \xi^2}{\tau}((a - \alpha \xi)(a + \alpha \xi)) = 0,$$  \hspace{1cm} (2.29)

which implies

$$a = \alpha \xi \text{ or } a = -\alpha \xi.$$  \hspace{1cm} (2.30)

If $a = \alpha \xi$, we can obtain from the first equation of (2.28) that

$$\delta \beta \xi^2 = 0,$$

which contradicts the assumptions of the coefficients in (1.1).
 Otherwise \( a = -\alpha \xi \). We can easily prove that it contradicts the assumptions of the coefficients in (1.1) in the same way. 

Thus we have completed the proof of Lemma 2.3. \( \Box \)

**Proposition 2.3.** There exist two positive constants \( C_1 \) and \( C_2 \) such that the solution \( V = V(t, \xi) \) to the Cauchy problem (1.2) satisfies in \( \{ \sigma \leq |\xi| \leq N \} \) the following estimate:

\[
|V(t, \xi)| \leq C \exp(-C_2 t)|V_0(\xi)|. \tag{2.31}
\]

**Proof.** From (2.25), we know \( \text{Re} \nu_k \geq C > 0 \) for all \( \xi \in \{ \sigma \leq |\xi| \leq N \} \). Let us consider the equation \( \partial_t V + BV = 0 \) with \( B = i\xi A_1 + A_0 \). Obviously there exists an invertible matrix \( L = L(\xi) \) satisfying \( LBL^{-1} := A \), a Jordan matrix. Thus \( W = LV \) satisfies

\[
\partial_t W + AW = 0, \tag{2.32}
\]

where the Jordan form of \( A \) depends on the multiplicities of the characteristic roots \( \nu_k(\xi) \) of the matrix \( B \) in \( \{ \sigma \leq |\xi| \leq N \} \).

If all roots are simple, then (2.31) follows immediately from (2.25). Otherwise, let us consider, for example, the case of two multiplicity of the characteristic roots \( \nu_1 = \nu_2, \nu_3 = \nu_4, \) but \( \nu_1 \neq \nu_3 \) at a point \( \xi_0 \in \{ \sigma \leq |\xi| \leq N \} \). Then we have

\[
A = \begin{pmatrix}
\nu_1(\xi_0) & 0 & 0 & 0 \\
1 & \nu_1(\xi_0) & 0 & 0 \\
0 & 0 & \nu_3(\xi_0) & 0 \\
0 & 0 & 1 & \nu_3(\xi_0)
\end{pmatrix}. \tag{2.33}
\]

Letting \( W = (W_1, W_2, W_3, W_4)^T \), we can obtain from (2.33) that

\[
\begin{aligned}
\partial_t W_1 + \nu_1(\xi_0)W_1 &= 0, \\
\partial_t W_2 + \nu_1(\xi_0)W_2 + W_1 &= 0, \\
\partial_t W_3 + \nu_3(\xi_0)W_3 &= 0, \\
\partial_t W_4 + \nu_3(\xi_0)W_4 + W_3 &= 0,
\end{aligned} \tag{2.34}
\]

with the following initial conditions

\[
W_k(0, \xi_0) = e_k(\xi_0) = \sum_{p=1}^{4} l_{kp}(\xi_0) V_{0,p}(\xi_0) \quad (1 \leq k \leq 4), \tag{2.35}
\]

where \( l_{kp} \) represents the \( p \)th component of the \( k \)th row of the matrix \( L \), \( V_{0,p} \) the \( p \)th component of the vector \( V_0(\xi_0) \).

The first and the third equations in (2.34) imply

\[
\begin{aligned}
W_1(t, \xi_0) &= \exp(-\nu_1(\xi_0)t)e_1(\xi_0), \\
W_3(t, \xi_0) &= \exp(-\nu_3(\xi_0)t)e_3(\xi_0).
\end{aligned} \tag{2.36}
\]

From (2.34) and (2.36), we can easily obtain

\[
\begin{aligned}
W_2(t, \xi_0) &= \exp(-\nu_1(\xi_0)t)(e_2(\xi_0) - e_1(\xi_0)t), \\
W_4(t, \xi_0) &= \exp(-\nu_3(\xi_0)t)(e_4(\xi_0) - e_3(\xi_0)t). \tag{2.37}
\end{aligned}
\]
Using the backward transformation from \( W \) to \( V \), we can obtain from (2.25), (2.36), and (2.37) that:

\[
|V(t, \xi_0)| \leq C_1(\xi_0) \exp(-C_2t)|V_0(\xi_0)|, \tag{2.38}
\]

where \( C_2 \in (0, C) \) (\( C \) is the constant from (2.25)).

For any \( \xi_0 \in \{\sigma \leq |\xi| \leq N\} \), we can derive (2.38) by using the transformation to the Jordan form as the previous procedure. Now it remains to verify (2.31) from (2.38) with the help of the compactness of \( \{\sigma \leq |\xi| \leq N\} \).

Let us consider

\[
\partial_t V + B(\xi)V = 0, \tag{2.39}
\]

where \( \xi \) is in a small neighborhood of \( \xi_0 \).

Obviously, (2.39) can be rewritten as

\[
\partial_t V + B(\xi_0)V = -(B(\xi) - B(\xi_0))V. \tag{2.40}
\]

Denote by \( \chi = \chi(t, s, \xi_0) \) the fundamental matrix of the operator \( \partial_t + B(\xi_0) \). In terms of (2.38) we know that \( \chi \) solves

\[
\partial_t \chi(t, s, \xi_0) + B(\xi_0)\chi(t, s, \xi_0) = 0, \quad \chi(s, s, \xi_0) = I,
\]

satisfying

\[
|\chi(t, s, \xi_0)| \leq C_1(\xi_0) \exp(-C_2(t-s)) \quad \text{for all } 0 \leq s \leq t. \tag{2.41}
\]

By Duhamel’s principle and Gronwall’s inequality, we can obtain from (2.40) and (2.41) with

\[
|B(\xi) - B(\xi_0)| < \varepsilon \quad \text{when } \xi \text{ is in a small neighborhood of } \xi_0,
\]

that

\[
|V(t, \xi)| \leq C_1(\xi_0) \exp(-C_2t) \exp(C_1(\xi_0)\varepsilon t)|V_0(\xi)|. \tag{2.42}
\]

Now choosing a suitable \( \varepsilon \) such that \( C_1(\xi_0)\varepsilon < \frac{C_2}{2} \), we can obtain the result of (2.31) for all \( \xi \) in the small neighborhood of \( \xi_0 \). Using the compactness of \( \{\sigma \leq |\xi| \leq N\} \), we can prove that (2.31) holds for all \( \xi \in \{\sigma \leq |\xi| \leq N\} \). This completes the proof of Proposition 2.3.

3. Proof of Theorems 1.1 and 1.2. In this section we shall prove Theorems 1.1 and 1.2.

Proof of Theorem 1.1. By considering the effect of frequency in different regions of phase space, we have found a method to diagonalize the system (1.2) and, furthermore, have obtained the results of Propositions 2.1, 2.2, and 2.3 in §2. By considering Propositions 2.1 through 2.3, we know that Theorem 1.1 is a direct consequence of these propositions. □

In the following procedure we will try to prove Theorem 1.2, namely, to derive the \( L^p - L^q \) decay estimate for the solution to (1.1). This is the central purpose of this paper. In order to do so, we first derive the \( L^2 - L^2 \) decay estimate and successively the \( L^1 - L^\infty \) decay estimate. Then by using the interpolation theorem, we obtain the result of Theorem 1.2.
First, we can easily obtain from Theorem 1.1,
\[ \|V(t, \xi)\|_{L^1(R)} \leq C\|V_0(\xi)\|_{L^2(R)} \]  \hfill (3.1)
where \( C \) represents a positive constant.

Subsequently, we will derive the \( L^1 - L^\infty \) decay estimate in different regions of phase space respectively.

Letting \( \phi(\xi) \in C_c^\infty(R) \) and \( \psi(\xi) \in C_c^\infty(R) \) such that
\[
\phi(\xi) = \begin{cases} 
1, & |\xi| \leq \sigma, \\
0, & |\xi| > \sigma + 1,
\end{cases} \quad \text{and} \quad \psi(\xi) = \begin{cases} 
1, & |\xi| \geq N + 1, \\
0, & |\xi| < N,
\end{cases}
\]
then we have the following.

**Proposition 3.1.** Let \( V(t, \xi) \) be the solution to the Cauchy problem (1.2). We know that \( V(t, \xi) \) has the following \( L^1 - L^\infty \) decay estimates
\[
\|F^{-1}(\phi(\xi)V(t, \xi))\|_{L^\infty(R)} \leq C(1 + t)^{-\frac{1}{2}}\|F^{-1}(\phi(\xi)V_0(\xi))\|_{L^1(R)} \quad \text{as} \quad |\xi| \leq \sigma \ll 1, \quad \|F^{-1}((1 - \phi(\xi))V(t, \xi))\|_{L^\infty(R)} \leq C\exp(-C_2t)\|F^{-1}((1 - \phi(\xi))\langle \xi \rangle^2V_0(\xi))\|_{L^1(R)} \quad \text{as} \quad |\xi| > \sigma \]  \hfill (3.2)
for \( \langle \xi \rangle^2 := 1 + |\xi|^2 \).

**Proof:** From Proposition 2.1, we obtain
\[
V(t, \xi) = (I + K_1 \xi)^{-1}K_2^{-1}(I + K_3 \xi^2)^{-1}(I + K_4 \xi)^{-1}M_1^{-1}\text{diag}\{\exp(-\nu_1(\xi)t), \exp(-\nu_2(\xi)t), \\
\exp(-\nu_3(\xi)t), \exp(-\nu_4(\xi)t)\}M_1(I + K_4 \xi)(I + K_3 \xi^2)K_2(I + K_1 \xi)V_0(\xi),
\]  \hfill (3.4)
for \( |\xi| \leq \sigma \).

Setting \( V = (V_1, V_2, V_3, V_4)^T \) and \( V_0 = (V_{1,0}, V_{2,0}, V_{3,0}, V_{4,0})^T \), we obtain for \( 1 \leq k \leq 4 \) the relations
\[
V_k(t, \xi) = \sum_{r,l=1}^4 \exp(-\nu_r(\xi)t)C_{rlk}(\xi)V_{l,0}(\xi)
\]  \hfill (3.5)
for \( |\xi| \leq \sigma \).

From the properties of \( M_1, \xi K_1, K_2, \xi^2 K_3, \) and \( \xi K_4 \), we know that \( C_{rlk}(\xi) \) tends to a constant \( C_{rlk}^0 \) if \( \xi \to 0 \). Obviously, the function \( C_{rlk}(\xi) \) is bounded for \( |\xi| \leq \sigma \).

Considering Lemma 2.1, we have
\[
\|F^{-1}(\phi(\xi)V_k(t, \xi))\|_{L^\infty(R)} \leq C\|\phi(\xi)V_k(t, \xi)\|_{L^1(R)} \leq C\sum_{r,l=1}^4 \|\exp(-\nu_r(\xi)t)C_{rlk}(\xi)\phi(\xi)V_{l,0}(\xi)\|_{L^1(R)} \leq C(1 + t)^{-\frac{1}{2}}\|\phi(\xi)V_{l,0}(\xi)\|_{L^\infty(R)} \leq C(1 + t)^{-\frac{1}{2}}\|F^{-1}(\phi(\xi)V_0(\xi))\|_{L^1(R)}.
\]  \hfill (3.6)
This completes the proof of (3.2).

Subsequently we derive the inequality (3.3).
Similarly, we obtain from Proposition 2.2 that
\[ V_k(t, \xi) = \sum_{r,l=1}^{4} \exp(-\nu_r(\xi)t)C_{rlk}(\xi)V_{i,0}(\xi) \]  
(3.7)
where \( C_{rlk}(\xi) \) tends to a constant \( C_{rlk}^0 \) if \(|\xi| \to \infty\) and the function is bounded for \( |\xi| \geq N \).

Taking into account of Lemma 2.2, we have for a large enough \( N \) the following estimate:
\[ \|F^{-1}(\psi(\xi)V_k(t, \xi))\|_{L^\infty(R)} \leq C\exp(-C_2t)\|\psi(\xi)V_0(\xi)\|_{L^1(R)}, \]
\[ \leq C\exp(-C_2t)\|F^{-1}(\psi(\xi)^2V_0(\xi))\|_{L^1(R)}. \]
(3.8)

Similarly, we can obtain from Proposition 2.3 and Lemma 2.3 for \( |\xi| \leq N \) that
\[ \|F^{-1}((1 - \phi(\xi) - \psi(\xi))V(t, \xi))\|_{L^\infty(R)} \leq C\exp(-C_2t)\|F^{-1}((1 - \phi(\xi) - \psi(\xi))^2V_0(\xi))\|_{L^1(R)}. \]
(3.9)

Summarizing (3.8) and (3.9), we conclude that Proposition 3.1 holds.

Now we will prove Theorem 1.2 in the following procedure.

Proof of Theorem 1.2. First from (3.1) we can obtain, by using the Parseval identity, that
\[ \|F^{-1}(V(t, \xi))\|_{L^2(R)} \leq C\|F^{-1}(V_0(\xi))\|_{L^2(R)}. \]
(3.10)

Noticing that
\[ V(t, \xi) = \phi(\xi)V(t, \xi) + \psi(\xi)V(t, \xi) + (1 - \phi(\xi) - \psi(\xi))V(t, \xi), \]
we obtain from (3.2) and (3.3) that
\[ \|F^{-1}(V(t, \xi))\|_{L^\infty(R)} \leq C(1 + t)^{-\frac{1}{2}}\|F^{-1}((\xi)^2V_0(\xi))\|_{L^1(R)}. \]
(3.11)

Using the interpolation theorem between (3.10) and (3.11), we obtain
\[ \|F^{-1}(V(t, \xi))\|_{L^q(R)} \leq C(1 + t)^{-\frac{1}{2}}\|F^{-1}((\xi)^2V_0(\xi))\|_{W^{-\frac{1}{2},q}(R)}, \]
(3.12)

where \( N \geq (1 - \frac{3}{2}) \frac{1}{q} + \frac{1}{q} = 1, 2 \leq q \leq \infty \) and \( C \) is a positive constant.

Theorem 1.2 is proved.

References


