算法图论

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The Seven Bridges of Königsberg

As far as the problem of the seven bridges of Königsberg is concerned, it can be solved by making an exhaustive list of all possible routes, and then finding whether or not any route satisfies the conditions of the problem. Because of the number of possibilities, this method of solution would be too difficult and laborious, …… – L. Euler, 1736

Finally, we note that the comments in 3., concerning the computational complexity of the problem solved, sound remarkably modern for a paper written in Latin in 1736. – B. Reed [30, Preface]
Dodecahedron and Icosahedron

Icosian Game [8]

Find paths and cycles on the dodecahedron graph satisfying various conditions. For example, for any length four path in the graph, find a Hamilton cycle containing this path.
Icosian Game [8]

Hamilton first communicated the Icosian Calculus in a letter dated 7 October 1856 and then used it as the basis for his new puzzle, the Icosian game. He exhibited this game at the meeting of the British Association at Dublin and sold the idea to a dealer in London for 25 pounds in 1857. The game was marketed in 1859, accompanied by a printed leaflet of instructions (challenges for the game players).
Sir William Rowan Hamilton (1805 - 1865) [17]

For the graph/group theory heritage of Sir William Rowan Hamilton, see [5, 8, 9, 17, 32].

The rotation symmetry group of the dodecahedron is $A_5$, the minimum non-solvable group. This group, which just corresponds to the Icosian Calculus, has itself as its commutator group and this is the reason, as known already by Evarist Galois (1811–1832), why the equation $x^5 - x + a = 0$ is not solvable in radicals [22, Lecture 5].

The six lines connecting the centers of opposite faces of the dodecahedron are equiangular and simple linear algebra will show you that there are no seven equiangular lines in $R^3$ [27, Miniature 9].
Graph Symmetry and Icosian Game

Serpens explores his world [32, Chap. 6]

The beautiful slides is available at:
http://www.cs.berkeley.edu/~sequin/TALKS/Banff05_HamSymm.ppt
On September 12, 1740, a Berlin mathematician Phillip Naudé posed this question to Leonhard Euler (1707–1783): In how many ways can a positive integer $N$ be written as a sum of positive integers? Euler finally solved the problem in a letter to Goldbach dated June 9, 1750 and the relevant paper was published in 1760 [2, 7, 15, 33].
One of the downsides of how we learn math in high school is that many of us come to believe that if we can’t solve a problem in ten or twenty minutes, then we can’t solve it at all. – K.P. Bogart [12]

The fourth pentagonal number is \( \frac{4(3 \times 4 - 1)}{2} = 22 \), which is the sum of the fourth square number \( 4^2 = 16 \) and the third triangular number \( \frac{3 \times (3 - 1)}{2} = 6 \).
Pentagonal Number Theorem

The Euler function $\phi$: $\phi(q) = \prod_{n=1}^{\infty} (1 - q^n) = 1 - q - q^2 + q^5 + q^7 - q^{12} - q^{15} \ldots = \sum_{n=-\infty}^{\infty} (-1)^n q^{\frac{n(3n-1)}{2}}$

The exponents make up a progression of a simple nature. This became immediately apparent to Euler after writing down some 20 terms; quite possibly he calculated about a hundred. He very reasonably says, "this is quite certain, although I cannot prove it;" ten years later he does prove it. He could not possibly guess that both series and product are part of the theory of elliptic modular functions. It is another tie-up between number-theory and elliptic functions. – Andrew Weil [34, pp. 97-98].
Let $p(n)$ be the number of partitions of $n$. The question of Naudé is to calculate $p(n)$. Consider the generating function

$$p(q) = \sum_{n=0}^{\infty} p(n) q^n.$$ 

It clearly holds

$$p(q) \phi(q) = 1.$$ 

This together with the Pentagonal Number Theorem gives a recursive formula for the function $p(n)$ and hence we have a simple algorithm to solve the problem of Naudé.
Let $k$ be the number of rows, let $\ell$ be the length of the shortest row, let $d$ be the length of the rightmost south-westerly diagonal and let $b$ be the length of the second rightmost south-westerly diagonal.
The Pentagonal Number Theorem can be restated as

$$E(n) - O(n) = \begin{cases} (-1)^k, & \text{if } n = \frac{k(3k \pm 1)}{2}, \\ 0, & \text{else,} \end{cases}$$

where $E(n)$ is the number of partitions of $n$ into an even number of unequal parts and $O(n)$ is the number of partitions of $n$ into an odd number of unequal parts. Note that $E(n) + O(n) = |D(n)|$ where $D(n)$ denote the set of partitions of $n$ into distinct parts.

Franklin introduced the following two maps on partitions which change the parity of the number of parts:

$\sigma$: move the shortest row to be a new rightmost diagonal

$\tau$: move the rightmost diagonal to be the new shortest row
\[ \mathcal{F} = \{ F \in \mathcal{D}(n) : d(F) \geq \ell(F) \}; \mathcal{F}' = \{ F' \in \mathcal{D}(n) : d(F') < \ell(F') \} \]

Note that \( \sigma(F) = F' \), \( \tau(F') = F \) in the above two cases. It remains to consider \( \{ F : k(F) = d(F) = \ell(F), F \in \mathcal{F} \} \) and \( \{ F' : d(F') = b(F'), (\ell(F') - d(F')) + (k(F') - b(F')) \leq 1, F' \in \mathcal{F}' \} \).
\{ F : k(F) = d(F) = \ell(F), F \in \mathcal{F} \} \neq \emptyset \text{ iff it contains a unique element, which looks like}

(k = 4) :

and so iff \( n = \frac{k(3k-1)}{2} \).

\{ F' : d(F') = b(F'), (\ell(F') - d(F')) + (k(F') - b(F')) \leq 1, F' \in \mathcal{F}' \} \neq \emptyset \text{ iff it contains a unique element, which looks like}

(k = 4) :

and so iff \( n = \frac{k(3k+1)}{2} \).
Lots of deep mathematics turn out to be related to the Euler function and its associated structure [1, 2, 4, 7, 15, 20, 33, 34]. Here is a very accessible report on the relevant dramatic and beautiful story: [22, Lecture 3: On collecting like terms, on Euler, Gauss, and MacDonald, and on missed opportunities].

Dyson is involved in studying the formula for powers of the Euler function $\phi$. He wrote the following regarding the opportunity which MacDonald picked and he missed in figuring out some exciting mathematics about powers of $\phi$ [20]:

"So I missed the opportunity of discovering a deeper connection between modular forms and Lie algebras, just because the number theorist Dyson and the physicist Dyson were not speaking to each other".
One true lesson to be drawn is the importance of studying the work of the masters, even those like Euler who died more than two centuries ago. The cleverness of Euler’s unpublished, second proof of the pentagonal number theorem allows us to discern facets of the Rogers-Fine identity that had gone unnoticed. – G.E. Andrews and J. Bell [2]
Problem: \( k \)-Disjoint Rooted Paths (or \( k \)-DRP)

Instance: A graph \( G \), and two sets of vertices \( S = \{ s_1, \ldots, s_k \} \) and \( T = \{ t_1, \ldots, t_k \} \).

Question: Are there \( k \) vertex disjoint paths \( P_1, \ldots, P_k \) from \( S \) to \( T \) in \( G \) such that \( P_i \) has endpoints \( s_i \) and \( t_i \)?

For any fixed positive integer \( k \), building on seminal work of Robertson and Seymour, Bruce Reed describes a nearly linear time (i.e. \( n \log(n) \)) algorithm for this problem in [30]. It shows how to build and use a decomposition tree for graphs excluding a fixed graph \( H \) as a minor and presents a linear time algorithm for testing if an input graph contains a fixed \( H \) as a minor.
Advances in computing power permit the study of huge networks. Much of the current research in graph theory focuses on how to handle such networks. We focus on three recent theoretical breakthroughs: The proof of the Strong Perfect Graph Conjecture, The Graph Minors Project, and Szemeredi’s Regularity Lemma. The bulk of the course will be devoted to the Graph Minors Project. The text for this part of the course will be a monograph written by the instructor.
Bruce Reed will visit Shanghai during June 16 – June 25, 2013 and will give three to five lectures on the Graph Minors Project based on his new monograph [30]. We intend to find more qualified audience (hopefully, many of you in this classroom will be present then) for these lectures.


http://en.wikipedia.org/wiki/Bruce_Reed_(mathematician)
There are 35 items in Part V (Theorems and Problems) of the book
One of the items is The Robertson-Seymour Theorem, prepared by Bruce Reed.

Our goal in this last chapter is a simple theorem, one which dwarfs any other result in graph theory and may doubtless be counted among the deepest theorems that mathematics has to offer: in every infinite set of graphs there are two such that one is a minor of the other. – R. Diestel [19, p. 333], http://www.math.uni-hamburg.de/home/diestel/books/graph.theory/preview/Ch12.pdf
Graph Minors and Tree Decompositions

- http://en.wikipedia.org/wiki/Tree_width
- http://en.wikipedia.org/wiki/Graph_structure_theorem
- http://en.wikipedia.org/wiki/Robertson%E2%80%93Seymour_theorem
- [16, 18, 19, 26, 29, 30] ...
Constraint Satisfaction Problem (CSP)

Figure: Sudoku

CSP is a one instance of the so-called Sum-of-Products class of problems [3] which we describe below.

Let $V$ be a set of discrete valued variables $\{X_1, \ldots, X_n\}$, $F$ be a set of functions $\{f_1, \ldots, f_m\}$ where each $f_i$ is defined over $E_i \subseteq V$, let $\bigoplus$ be an addition operator and $\bigotimes$ be a multiplication operator. We want to efficiently calculate

$$\bigoplus_{X_1} \cdots \bigoplus_{X_n} \bigotimes_{i=1}^{m} f_i(E_i).$$

It turns out that a critical measure of the complexity of this task is the width of the hypergraph $(V, E)$ where $E = \{E_1, \ldots, E_m\}$.
A basic observation is that complexity of computational problems often depends on structural parameters of the components, i.e. the participating objects, of the problem. To efficiently solve problems of practical interest, it is necessary to study possible parameters and structural properties of the participating objects and their influence on the complexity of the considered problems.

With respect to the previous observations, the most prominent classes of structures are those closely related to trees.

– P. Hiliněny, S-I. Oum, D. Seese, G. Gottlob [23]
Maria Chudnovsky, IEOR 8100 Graph Theory

This course will be a survey of various width parameters in graph theory, such as path-width, tree-width, rank-width, clique-width etc. The focus will be on sufficient conditions for a class of graphs to have bounded width (under the different definitions), and on algorithmic implications of bounded width. We will also discuss similar concepts for tournaments (these are orientations of complete graphs, and over the last few years substantial progress has been made in that area).
What Do They Do There?


https://is.muni.cz/course/fi/spring2013/MA052

http://orion.math.iastate.edu/rymartin/
ISU608EGT/S12/ISU608S12.html

https://www.math.lsu.edu/~kearney/
4171spring2012/project.html

http://www2.informatik.hu-berlin.de/logik/lehre/WS11-12/Para/index.html
Should You Stay in This Course?

- There are much beautiful mathematics hidden behind a good algorithm. It is the development of relevant mathematics but not a specific programming language that plays key role in settling an algorithmic problem. I feel ashamed that I know little about programming and so please do not come for learning programming.
- In most part of this course I intend to study together with you some width parameters and some low-width structures. Have a look at, say [10, 11, 13, 14, 16, 18, 19, 21, 23, 26, 28, 29, 30].
- I hope that you know some basic linear algebra and have an open brain.
- Do not worry about failing to catch all my words in the classroom. I assure you I have far greater problem in understanding some simple mathematics, either written down by others or not. If you are really motivated, you just keep moving forward and wait for the final moment.
在我看来，尊严首先是智力的尊严。很长一段时间了，这个民族首先失去智力上的尊严。... 比起思维的结果，思维本身就是一种尊严。只是总有人放弃了这过程。... 我真正认为，才华正是来自于尊严。— 李承鹏 [24, 尊严]
William Tutte (1917-2002) studied at Cambridge where his fascination for mathematical puzzles brought him into contact with like-minded undergraduates, together becoming known as the 'Trinity four', the founders of modern graph theory. His notable problem-solving skills meant he was brought to Bletchley Park during World War Two. Key in the enemy codebreaking efforts, he cracked the Lorenz cipher for which the Colossus machine was built, making his contribution comparable to Alan Turing’s codebreaking for Enigma. Following his incredible war effort Tutte returned to academia and became a fellow of the Royal Society in Britain and Canada, finishing his career as Distinguished Professor Emeritus at the University of Waterloo, Ontario.
Squaring the Square: The First Mathematical Work by the ‘Trinity Four’ (Blanche Descartes)

Fig. 20.10. The unique perfect squared square of smallest order

[13, p. 549]
In 1884, the physicist Peter Tait proposed the following conjecture: Every 3-connected planar cubic graph has a Hamiltonian cycle.

It is known that Tait’s Conjecture implies the Four Color Conjecture and that is the real motivation of Tait in making his conjecture. (The Hamiltonian cycle of the map divides the sphere into two regions and the countries in each region can be colored with two colors.)

http://en.wikipedia.org/wiki/Tait%27s_conjecture
Tutte (1946) [31, 32]: If every 3-connected planar cubic graph has a Hamiltonian cycle, then for any two adjacent edges of a 3-connected planar cubic graph $G$, we can find a Hamiltonian cycle of $G$ passing through the two edges.
Proof by Contradiction

Assume that $G$ is a 3-connected planar cubic graph, that $G$ has a vertex $v$ and three edges $A$, $B$, $C$ incident to $v$ and that $G$ has no Hamiltonian cycle passing through $B$ and $C$. 

![Diagram showing a vertex v and edges A, B, C incident to v, with a shaded region M.](image)
This implies that there is no $v_B, v_C$-path in the following graph which pass through all vertices in $M$ without repetition.

Figure: [32, Fig. 2.2].
So, each Hamiltonian cycle of the next graph, which must exist according to the assumption, should pass through $A_1, A_2$ and $A_3$. This is impossible.

Figure : [32, Fig. 2.3].
Tutte (1946) [31, 32]: If every 3-connected planar cubic graph has a Hamiltonian cycle passing through any two adjacent edges, then for any planar embedding of a 3-connected planar cubic graph $G$ and any two edges bordering the same face of the embedding, $G$ has a Hamiltonian cycle passing through the two edges.

Figure : [32, Fig. 2.4].

Figure : A Hamiltonian cycle passing through $A_1, A_2$ must also pass through $B_1, B_2$. 
Tutte (1946) [31, 32]: If every 3-connected planar cubic graph has a Hamiltonian cycle, then for any planar embedding of a 3-connected planar cubic graph $G$ and any two radiants of a face of the embedding, $G$ has a Hamiltonian cycle passing through the two radiants.
Proof

Figure: $Y$-$\Delta$ transformation [32, Fig. 2.5].
A Counterexample

The pentagonal prism shown below has no Hamiltonian cycle passing through the two brown edges.

![Figure: [32, Fig. 2.1]]

This allows Tutte to assert that Tait’s Conjecture is wrong.
Barnette’s Conjecture [6]: Every 3-connected cubic planar bipartite graph is Hamiltonian [13, p. 481].
A separating triangle of a map on the sphere is a cycle of three edges that is not the boundary of a face.

Hassler Whitney (1931) [35]: Any map on the sphere without separating triangles and with all faces being triangles must have a Hamiltonian cycle.

P. Li, Y. Wu, Hamiltonian connectedness and thick Hamiltonian orderings of graphs and interval graphs.

http://math.sjtu.edu.cn/faculty/ykwu/SHU201303.pdf
讨论班

图论讨论班，每周二晚上七点，数学楼1106教室，以中文为工作语言。

坂内英一教授本学期继续组织代数组合讨论班，以英文为工作语言，时间地点待定，有兴趣者请与他联系\(^1\)。

欢迎参加。

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无地自容

人潮人海中又看到你
一样迷人一样美丽
曾感到过寂寞也曾被别人冷落
却从未有感觉我无地自容
窦唯，《无地自容》

Figure：窦唯

http://v.ku6.com/show/0oLQNzY2-J1BBPtG.html?loc=tashangchuan


